Higher-order time-delay interferometry

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Time-delay interferometry (TDI) is the data processing technique that cancels the large laser phase fluctuations affecting the one-way Doppler measurements made by unequal-arm space-based gravitational wave interferometers. In a previous publication we derived TDI combinations that *exactly* cancel the laser phase fluctuations up to first order in the interspacecraft velocities. This was done by interfering two digitally synthesized optical beams propagating a number of times clockwise and counterclockwise around the array. Here we extend that approach by showing that the number of loops made by each beam before interfering corresponds to a specific higher-order TDI space. In it the cancellation of laser noise terms that depend on the acceleration and higher-order time derivatives of the interspacecraft light-travel times is achieved exactly. Similarly to what we proved for the second-generation TDI space, elements of a specific higher-order TDI space can be obtained by first "lifting" the basis (α, β, γ, X) of the first-generation TDI space to the higher-order space of interest and then taking linear combinations of them with coefficients that are polynomials of the six delays operators. Higher-order TDI might be required by future interplanetary gravitational wave missions whose interspacecraft distances vary appreciably with time, in particular, relative velocities are much larger than those of currently planned arrays.

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I. INTRODUCTION

Interferometric detectors of gravitational waves may be thought of as optical configurations with one or more arms folding coherent trains of light. At points where these intersect, relative fluctuations of frequency or phase are monitored (homodyne detection). Interference of two or more beams, produced and monitored by a nonlinear device such as a photo detector, exhibits sidebands as a low frequency signal. The observed low frequency signal is due to frequency variations of the sources of the beams about the nominal frequency ν_0 of the beams, to relative motions of the sources and any mirrors (or optical transponders) that do any beam folding, to temporal variations of the index of refraction along the beams, and, according to general relativity, to any time-variable gravitational fields present, such as the transverse traceless metric curvature of a passing plane gravitational wave train. To observe gravitational waves in this way, it is thus necessary to control, or monitor, the other sources of relative frequency fluctuations, and, in the data analysis, to optimally use algorithms based on the different characteristic interferometer responses to gravitational waves (the signal) and on the other sources (the noise).

By comparing phases of split beams propagated along equal but nonparallel arms, frequency fluctuations from the source of the beams are removed directly at the photo detector and gravitational wave signals at levels many orders of magnitude lower can be detected. Especially for interferometers that use light generated by presently available lasers, which display frequency stability roughly a few parts in 10^{-13} in the millihertz band, it is essential to remove these fluctuations when searching for gravitational waves of dimensionless amplitude smaller than 10^{-21}.

Space-based, three-arm interferometers [1–5] are prevented from canceling the laser noise by directly interfering the beams from their unequal arms at a single photo detector because laser phase fluctuations experience different delays. As a result, the Doppler data from the three arms are measured at different photo detectors on board the three spacecraft and are then digitally processed to compensate for the inequality of the arms. This data processing technique, called time-delay interferometry (TDI) [6], entails time-shifting and linearly combining the Doppler measurements so as to achieve the required sensitivity to gravitational radiation.

In a recent article [7] we reanalyzed the space of the TDI measurements that *exactly* cancel the laser noise up to the

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interspacecraft linear velocity terms, i.e. the so-called second-generation TDI space. By first regarding the basis $(\alpha, \beta, \gamma, X)$ of the first-generation TDI space as the result of the interference of two synthesized light beams propagating once, clockwise and counterclockwise around the array, we then showed that exact cancellation of the laser noise terms containing the interspacecraft velocities could be achieved by making these beams complete a larger number of loops around the array before interfering. In the case of the Sagnac combinations (α, β, γ), the minimum number of loops made by each beam around the array to exactly cancel the laser noise linear velocity terms was found to be three, while for the unequal-arm Michelson combination, X, the minimum number of loops was equal to two. In physical terms, by making the synthesized beams go around the array in the clockwise and counterclockwise sense a number of times before interfering, one ends up averaging out the effects due to the rotation of the array and the time dependence of the interspacecraft light-travel times. Since the exact cancellation of the laser noise for any arbitrary time-dependent delays cannot be achieved [8], in this paper we prove that there exists a correspondence between the number of clockwise and counterclockwise loops made by the beams around the array and the order of cancellation of the laser noise in the kinematic terms of the interspacecraft light-travel times. In the case of the unequal-arm Michelson combination this result had already been noticed through a numerical analysis [9] and in this article we actually prove it analytically.

The paper is organized as follows. In Sec. II we review some of the results presented in [7] that are relevant here. We first summarize the "lifting" [7] technique, in which elements of a basis of the first-generation TDI space are rewritten in terms of the six delay operators. Then their corresponding second-generation and higher-order TDI expressions are obtained by acting on specific combinations of their data with uniquely identified polynomials of the six delays. This operation is key to our method as it allows us to generalize the main property of a basis of the first-generation TDI space: elements of the second-generation and higher-order TDI spaces are obtained by taking linear combinations of properly delayed lifted basis [7]. The higher-order TDI combinations cancel laser noise terms depending on the second- and higher-order time derivatives of the light-travel times. In physical terms, the operation of lifting corresponds to two light beams making clockwise and counterclockwise loops around the array before being recombined on board the transmitting spacecraft. In so doing the time variations of the light-travel times is averaged out more and more accurately. As an exemplification, after applying an additional lifting procedure to the second-generation TDI combinations $(\alpha_2, \beta_2, \gamma_2, X_2)$ derived in [7], we obtain the corresponding combinations $(\alpha_3, \beta_3, \gamma_3, X_3)$. In Sec. III, after deriving useful identities of the six delay operators, we mathematically prove that $(\alpha_3, \beta_3, \gamma_3, X_3)$ exactly cancel the laser noise up to terms quadratic in the interspacecraft velocities and linear in accelerations, and that higher-order TDI combinations cancel the laser noise up to higher-order time derivatives of the interspacecraft light-travel times. In Sec. IV we then present our comments on our findings and our conclusions.

II. THE LIFTING PROCEDURE

Here we present a brief summary of the lifting procedure discussed in [7]. There it was shown that the operation of lifting provides a way for deriving elements of the second-generation TDI space by lifting combinations of the first-generation TDI space. As it will become clearer below, the lifting procedure can be generalized so as to provide TDI combinations that exactly cancel the laser noise containing delays of any order arising from kinematics.

The one-way Doppler data y_i, $y_{i'}$ are written in terms of the laser noises using the notation introduced in [6,10]. We index the one-way Doppler data as follows: the beam arriving at spacecraft i has subscript i and is primed or unprimed depending on whether the beam is traveling clockwise or counterclockwise around the interferometer array, with the sense defined by the orientation of the array shown in Fig. 1. Because of the Sagnac effect due to the rotation of the array, the light-travel time from say spacecraft i to j is not the same as the one from j to i. Therefore $L_i \neq L'_i$ and so we have six unequal time-dependent time delays (we choose units so that the velocity of light c is unity and L_i, L'_i have dimensions of time—they are actually L_i/c, L'_i/c.). The corresponding delay operators are labeled as $\mathcal{D}_i$ and $\mathcal{D}_{i'}$ and are defined by their action on an arbitrary time series $\Psi(t)$ as $\mathcal{D}_i\Psi(t) \equiv \Psi(t - L_i)$ and $\mathcal{D}_{i'}\Psi(t) \equiv \Psi(t - L'_i)$ respectively.

The one-way phase measurements are then given by the following expressions [6]:

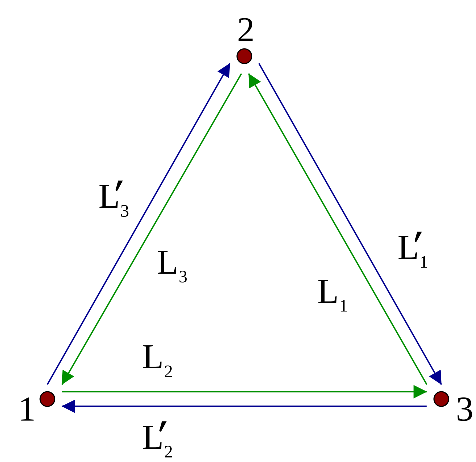

FIG. 1. Schematic diagram of the directed array. The spacecraft are labeled $i = 1, 2, 3$ while the optical paths are denoted by L_i, L'_i where the convention is that the index i corresponds to the opposite spacecraft.

$$
\begin{aligned}
y_1 &= \mathcal{D}_3 C_2 - C_1, & y_{1'} &= \mathcal{D}_{2'} C_3 - C_1, \\
y_2 &= \mathcal{D}_1 C_3 - C_2, & y_{2'} &= \mathcal{D}_{3'} C_1 - C_2, \\
y_3 &= \mathcal{D}_2 C_1 - C_3, & y_{3'} &= \mathcal{D}_{1'} C_2 - C_3.
\end{aligned} \tag{2.1}
$$

Thus, as seen in the figure, y_1 for example is the phase difference time series measured at reception at spacecraft 1 with transmission from spacecraft 2 (along L_3).[1]

As emphasized in [7], to generate elements of the second-generation TDI space with the lifting procedure one first needs to derive the expressions of the four generators, α, β, γ, X, of the first-generation TDI that include the six delays $i, i'i, i' = 1, 2, 3, 1', 2', 3'$. Since these combinations correspond to two beams propagating clockwise and counterclockwise once, the lifting procedure makes these beams propagate clockwise and counterclockwise a number of times before being made to interfere. The resulting data combinations exactly cancel the laser noise terms linear in the interspacecraft velocities. The lifting procedure is unique and can be applied iteratively an arbitrary number of times. As we will show below, each iteration suppresses the laser noise significantly more than that achieved at the previous iterative step. To be specific, a second-generation TDI combination cancels the laser noise up to linear velocity terms, while the corresponding third generation cancels it up to the acceleration and terms quadratic in velocities. It should be noticed that some elements of the second-generation TDI space, like the Sagnac combinations α, β, γ, require more than two "lifting" iterations to exactly cancel the laser noise up to the linear velocity terms [7]. Therefore we will refer to the nth-generation TDI space as those TDI combinations that *exactly* cancel the laser noise up to the $(n-1)$th time derivatives of the time delays. We emphasize that, although lifting allows us to established a homomorphism between the first-generation and any higher-order TDI space [7], it does *not* generate the entire higher-order TDI space of interest. This is because it does not construct the basis of such a space.

A. Time-varying arms' lengths and vanishing commutators

If the arms' lengths are time dependent, then the operators do not commute and the laser noise will not cancel. However, if the arms' lengths are analytic functions of time, we can Taylor expand the operators and keep terms to a specific order in the time derivatives of the light-travel times. Although in the case of the currently envisioned missions [1–5] it is sufficient to cancel terms that are only first order in $\dot{L}_i$ and $\dot{L}'_i$ or linear in velocities [6,11]. However, in future missions able to handle the detection of high interspacecraft beat notes, one may have to account for higher-order time derivatives of the interspacecraft distances. In those cases the lifting procedure presented in this article provides a method for obtaining TDI combinations that cancel the laser noise up to the order required.

Let us first start by noting the effect of n operators $\mathcal{D}_{k_1}, \ldots, \mathcal{D}_{k_n}$ applied on the laser noise $C(t)$. We also write the expressions in a neat form. For three operators we obtain[2]

$$
\begin{aligned}
\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3 C(t) &= C[t - L_3(t - L_2(t - L_1) - L_1) - L_2(t - L_1) - L_1] \\
&= C\Big[t - L_1 - L_2 - L_3 + (L_2 v_3 + L_1 v_2 + L_1 v_3) - L_1 v_2 v_3 - \frac{1}{2}(L_1 + L_2)^2 f_1 + L_1^2 f_2)\Big]. \\
&= C\left(t - \sum_{i=1}^{3} L_i + V_3 - Q_3 - F_3\right), \qquad (2.2) \\
&\approx C\left(t - \sum_{i=1}^{3} L_i\right) + (V_3 - Q_3 - F_3)\dot{C}\left(t - \sum_{i=1}^{3} L_i\right) + \frac{1}{2} V_3^2 \ddot{C}\left(t - \sum_{i=1}^{3} L_i\right), \qquad (2.3)
\end{aligned}
$$

where

$$
\begin{aligned}
V_3 &= L_1 v_2 + (L_1 + L_2) v_3, \\
Q_3 &= L_1 v_2 v_3, \\
F_3 &= \frac{1}{2}[L_1^2 f_2 + (L_1 + L_2)^2 f_3],
\end{aligned} \tag{2.4}
$$

[1] Besides the primary interspacecraft Doppler measurement y_i, $y_{i'}$ that contains the gravitational wave signal, other metrology measurements are made on board an interferometer's spacecraft. This is because each spacecraft is equipped with two lasers and two proof-masses of the onboard drag-free subsystem. It has been shown [6], however, that these onboard measurements can be properly delayed and linearly combined with the interspacecraft measurements to make the realistic interferometry configuration equivalent to that of an array with only three lasers and six one-way interspacecraft measurements.

[2] The operators could refer to either L_i or $L_{i'}$. We do not write the primes explicitly in order to avoid clutter but the identities that we derive hold in either case. Instead of writing $\mathcal{D}_{k_p}$ we have denoted the same by just $\mathcal{D}_p$ where p can take any of the values $1, 2, 3, 1', 2', 3'$.

where $v_i = \dot{L}_i$ and $f_i = \ddot{L}_i$. We have neglected higher-order terms of order $o(v^3)$, $o(vf)$ etc. while obtaining the above results. We have kept terms up to the quadratic order in velocities and linear in accelerations. We further denote by V, Q, F, the terms linear in velocities, quadratic in velocities, and linear in acceleration, respectively. For four operators $\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_4$ operating on $C(t)$, we obtain

$$\begin{aligned} V_4 &= L_1 v_2 + (L_1 + L_2)v_3 + (L_1 + L_2 + L_3)v_4, \\ Q_4 &= L_1[v_2 v_3 + v_2 v_4 + v_3 v_4] + L_2 v_3 v_4, \\ F_4 &= \frac{1}{2}[L_1^2 f_2 + (L_1 + L_2)^2 f_3 + (L_1 + L_2 + L_3)^2 f_4], \end{aligned} \tag{2.5}$$

with the expression of C being essentially the same as in Eq. (2.3) but V_3, Q_3, F_3 replaced by V_4, Q_4, F_4 etc. Also we find that there are recursion relations like $Q_4 = V_3 v_4 + Q_3$ which makes it convenient to derive the general expressions for n operators. Accordingly, the general expression for n operators is obtained from the above considerations by induction:

$$\begin{aligned} \mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\ldots\mathcal{D}_n C(t) &\approx C\left(t - \sum_{i=1}^{n} L_i\right) + (V_n - Q_n - F_n)\dot{C}\left(t - \sum_{i=1}^{n} L_i\right) + \frac{1}{2}V_n^2 \ddot{C}\left(t - \sum_{i=1}^{n} L_i\right), \\ V_n &= \sum_{i=1}^{n-1} L_i \left(\sum_{j=i+1}^{n} v_j\right), \\ Q_n &= \sum_{i=1}^{n-2} L_i \left(\sum_{j=i+1,k>j}^{n} v_j v_k\right), \\ F_n &= \frac{1}{2}\sum_{j=2}^{n} f_j \left(\sum_{i=1}^{j-1} L_i\right)^2. \end{aligned} \tag{2.6}$$

Let us interpret the right-hand side of this equation. The first term is just the laser noise at a delayed time that is equal to the sum of the delays at time t. If the arms' lengths were constant in time this would be the only term that would be present and would be sufficient to cancel the laser frequency noise. These are just the first generation TDI and the operators commute. The second term, on the other hand, involves the *multiplication* of $\dot{C}$ evaluated at the delayed time by an expression involving V, Q, F—it contains terms up to the second order in velocities and linear in accelerations. This term makes the operators noncommutative. The third term instead includes the second derivative of the laser noise and contains terms quadratic in velocities. As shown in [6,7,12] certain commutators cancel the laser noise up to linear velocity terms in the following general way:

$$[x_1 x_2 \ldots x_n, x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}] = 0, \tag{2.7}$$

where the "zero" on the right-hand side means up to first order in the linear velocity and σ is a permutation on the n symbols. However, as it will be shown in the next section, the expression on the left-hand side of Eq. (2.7) allows us to prove that, for a given n and a specific permutation of the indices, the cancellation of the laser noise achieved is up to the time derivatives of $(n-1)$th order in interspacecraft time delays.

Since this general result will be proved by induction, we first provide the expressions for the higher-order (third-generation TDI) Michelson and Sagnac combinations (α_3, β_3, γ_3, X_3) and show they can iteratively be related to their corresponding previous-order combinations.

B. The unequal-arm Michelson X_3

To derive the expression for X_3 we recall how the second-generation expression X_2 was derived [7,11]. The unequal-arm Michelson combination include only the four one-way Doppler measurements (y_1, $y_{1'}$, $y_{2'}$, y_3) from the two arms centered on spacecraft 1. They enter in X through the following synthesized two-way Doppler data:

$$\begin{aligned} X_\uparrow &\equiv y_1 + \mathcal{D}_3 y_{2'} = (\mathcal{D}_3\mathcal{D}_{3'} - I)C_1, \\ X_\downarrow &\equiv y_{1'} + \mathcal{D}_{2'} y_3 = (\mathcal{D}_{2'}\mathcal{D}_2 - I)C_1, \end{aligned} \tag{2.8}$$

where we included the expressions of their residual laser noises. These expressions imply the following residual laser noise in the first-generation TDI combination X [6,7]:

$$\begin{aligned} X &\equiv (\mathcal{D}_3\mathcal{D}_{3'} - I)X_\downarrow - (\mathcal{D}_{2'}\mathcal{D}_2 - I)X_\uparrow \\ &= [\mathcal{D}_3\mathcal{D}_{3'}, \mathcal{D}_{2'}\mathcal{D}_2]C_1 \equiv \mathcal{X}_1 C_1. \end{aligned} \tag{2.9}$$

Here we have defined the commutator $\mathcal{X}_1 = [\mathcal{D}_3\mathcal{D}_{3'}, \mathcal{D}_{2'}\mathcal{D}_2]$ as the first commutator which is associated with the first-generation unequal-arm Michelson combination. It is different from zero when the delays are functions of time and is linear in the interspacecraft relative velocities.

We now rewrite the above expression for X in terms of its two synthesized beams. They are equal to

$$X_{\uparrow\uparrow} \equiv \mathcal{D}_{2'}\mathcal{D}_2 X_\uparrow + X_\downarrow = (\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'} - I)C_1,$$
$$X_{\downarrow\downarrow} \equiv \mathcal{D}_3\mathcal{D}_{3'} X_\downarrow + X_\uparrow = (\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2 - I)C_1. \tag{2.10}$$

The expression of X_2 can then be derived by repeating the same procedure used for obtaining X. This results in the following combination:

$$X_2 \equiv (\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2 - I)X_{\uparrow\uparrow} - (\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'} - I)X_{\downarrow\downarrow}$$
$$= [\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2, \mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}]C_1 \equiv \mathcal{X}_2 C_1 = 0, \tag{2.11}$$

where we have defined the second commutator $\mathcal{X}_2 = [\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2, \mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}]$. The equality to zero, which means "up to terms linear in velocity," follows from the general property of the commutators of the delay operators proved in [7] and recalled in the previous section. This can also be understood from the following argument. Since we need to cancel terms only up to linear order in velocities for X_2, we only need to consider the quantities V_n of Eq. (2.6) for the commutator. Here $n = 8$ because we have a product of eight delay operators $\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}$ in the first term of the commutator. The explicit expression is

$$V_8 = L_3(3v_{3'} + 2v_{2'} + 2v_2 + v_3)$$
$$+ L_{3'}(2v_{2'} + 2v_2 + v_3 + v_{3'})$$
$$+ L_{2'}(3v_2 + 2v_3 + 2v_{3'} + v_{2'})$$
$$+ L_2(2v_3 + 2v_{3'} + v_{2'} + v_2). \tag{2.12}$$

A permutation of indices $3 \leftrightarrow 2'$, $3' \leftrightarrow 2$ produces the second term in the commutator. But under this permutation of indices as seen from Eq. (2.12) the quantity V_8 is invariant. Since the second term of the commutator has the opposite sign, the V terms cancel out to give zero.

Let us define $A_1 \equiv \mathcal{D}_3\mathcal{D}_{3'}$ and $B_1 \equiv \mathcal{D}_{2'}\mathcal{D}_2$. We have the following commutator's identity:

$$[A_1B_1, B_1A_1] = [[A_1, B_1], A_1B_1], \tag{2.13}$$

from which it follows that

$$\mathcal{X}_2 \equiv [\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2, \mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}] = [\mathcal{X}_1, \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2]. \tag{2.14}$$

Similar to what was done for both X and X_2, one can obtain X_3. From the expression for X_2 above we can write the following two combinations corresponding to two synthesized beams making three zero-area closed loops along the two arms of the array. We have

$$X_{\uparrow\uparrow\uparrow} \equiv \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2 X_{\uparrow\uparrow} + X_{\downarrow\downarrow}$$
$$= (\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'} - I)C_1, \tag{2.15}$$

$$X_{\downarrow\downarrow\downarrow} \equiv \mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'} X_{\downarrow\downarrow} + X_{\uparrow\uparrow}$$
$$= (\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2 - I)C_1, \tag{2.16}$$

which implies the following expression of the residual laser noise in X_3:

$$X_3 \equiv (\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2 - I)X_{\uparrow\uparrow\uparrow} - (\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'} - I)X_{\downarrow\downarrow\downarrow}$$
$$\equiv \mathcal{X}_3 C_1 = [\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2, \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}]C_1. \tag{2.17}$$

From the commutator identity derived earlier we see that $\mathcal{X}_3$ can be written in the following way:

$$\mathcal{X}_3 \equiv [\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2, \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}] = [\mathcal{X}_2, \mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2], \tag{2.18}$$

where $\mathcal{X}_2$ is in fact given by Eq. (2.14), the operator of the second-generation unequal-arm Michelson combination. We then conclude that the following identity is satisfied in general:

$$\mathcal{X}_n = [X_{n-1}, \mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2 \ldots], \tag{2.19}$$

where the total number of delay operators on the right-hand side is equal to 2^n, as one can easily infer.

In the following section we will return to the expression of X_3 and higher-order unequal-arm Michelson combinations. There we will show that X_3 cancels laser noise terms that are quadratic in the interspacecraft velocities and linear in the acceleration, and prove a general theorem by which TDI combinations of order n (such as X_n) cancel the laser noise up to $(n-1)$th time derivatives of the time delays.

C. The Sagnac combination α_3

A TDI Sagnac combination, α_n, represents the result of the interference of two synthesized light-beams on board spacecraft 1 after making an equal number of clockwise and counterclockwise loops around the array. In [7] we obtained the expression of α_2, the second-generation TDI Sagnac combination, that exactly cancels laser noise up to terms linear in the interspacecraft velocities. In what follows we derive α_3 by first recalling the expressions of α, $\alpha_{1.5}$, and α_2, and their residual laser noises

$$\alpha \equiv \alpha_\uparrow - \alpha_\downarrow = (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'})C_1, \tag{2.20}$$

where $\alpha_\uparrow$ and $\alpha_\downarrow$ are equal to the following combinations of the one-way heterodyne measurements [7]:

$$\begin{aligned} \alpha_\uparrow &\equiv y_1 + \mathcal{D}_3 y_2 + \mathcal{D}_3\mathcal{D}_1 y_3 = (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2 - I)C_1, \\ \alpha_\downarrow &\equiv y_{1'} + \mathcal{D}_{2'} y_{3'} + \mathcal{D}_{2'}\mathcal{D}_{1'} y_{2'} = (\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)C_1. \end{aligned} \tag{2.21}$$

The Sagnac combination $\alpha_{1.5}$ is then obtained by making the beams go around the array one additional time and results in the following expression:

$$\begin{aligned} \alpha_{1.5} &\equiv (\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)\alpha_\uparrow - (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2 - I)\alpha_\downarrow \\ &\equiv \sigma_{1.5}C_1 = [\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}, \mathcal{D}_3\mathcal{D}_1\mathcal{D}_2]C_1. \end{aligned} \tag{2.22}$$

From the properties of commutators derived in [7], we recognize that the right-hand side of Eq. (2.22) does not cancel the laser noise containing terms linear in the velocities. However, by making the beams go around the array one more time, we obtain the following expression of the second-generation Sagnac combination α_2:

$$\begin{aligned} \alpha_2 &= (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)\alpha_{\uparrow\uparrow} \\ &\quad - (\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2 - I)\alpha_{\downarrow\downarrow}, \\ &\equiv \sigma_2 C_1 = [\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}, \mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2]C_1. \end{aligned} \tag{2.23}$$

In Eq. (2.23) $\alpha_{\uparrow\uparrow}$, $\alpha_{\downarrow\downarrow}$ are equal to the following combinations of the six delay operators $\mathcal{D}_i, \mathcal{D}_j, i = 1, 2, 3, j = 1', 2', 3'$ [7]:

$$\begin{aligned} \alpha_{\uparrow\uparrow} &= \mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\alpha_\uparrow + \alpha_\downarrow \\ &= (\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2 - I)C_1, \\ \alpha_{\downarrow\downarrow} &= \alpha_\uparrow + \mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\alpha_\downarrow \\ &= (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)C_1. \end{aligned} \tag{2.24}$$

The operator in Eq. (2.23) acting on C_1 is the commutator of two delay operators, each containing the same number of primed and unprimed delay operators and related by permutations of their indices. We therefore conclude, from the commutator identities derived in the previous section, that the above expression results in the exact cancellation of the laser noise up to linear velocity terms.

Let us now consider the following two combinations entering in α_2:

$$\begin{aligned} \alpha_{\uparrow\uparrow\uparrow} &= \mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\alpha_{\uparrow\uparrow} + \alpha_{\downarrow\downarrow} \\ &= (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2 - I)C_1, \\ \alpha_{\downarrow\downarrow\downarrow} &= \mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\alpha_{\downarrow\downarrow} + \alpha_{\uparrow\uparrow} \\ &= (\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)C_1. \end{aligned} \tag{2.25}$$

From Eq. (2.25) above we obtain the following expression for α_3 and its residual laser noise:

$$\begin{aligned} \alpha_3 &= (\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)\alpha_{\uparrow\uparrow\uparrow} - (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2 - I)\alpha_{\downarrow\downarrow\downarrow} \\ &\equiv \sigma_3 C_1 = [\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}, \mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2]C_1. \end{aligned} \tag{2.26}$$

If we now define $A_1 \equiv \mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}$, $B_1 \equiv \mathcal{D}_3\mathcal{D}_1\mathcal{D}_2$, we see that the right-hand-side of Eq. (2.26) can be written as $[A_1B_1B_1A_1, B_1A_1A_1B_1]$, which is also equal to $[[A_1B_1, B_1A_1], A_1B_1B_1A_1]$ from the commutator's identity derived earlier. From these considerations we finally have,

$$\sigma_3 = [\sigma_2, \mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2]. \tag{2.27}$$

As in the case of the expression for the operator $\mathcal{X}_n$ derived in the previous section, here too we can relate the operator σ_n to the operator σ_{n-1} in the following way:

$$\sigma_n = [\sigma_{n-1}, \mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2 \ldots], \tag{2.28}$$

where the total number of delay operators on the right-hand side is equal to 3×2^n, as one can easily infer.

III. HIGHER-ORDER TDI

In the previous section we showed that an order-n TDI combination can be written in terms of its corresponding $(n-1)$-order one through a commutator identity [see Eqs. (2.19) and (2.28)]. In this section we will take advantage of this property by first proving that the third-order TDI combinations $\alpha_3, \beta_3, \gamma_3, X_3$ cancel the laser noise up to terms quadratic in the interspacecraft velocities and linear in the accelerations. We will then generalize this result and prove that combinations of order n cancel exactly the laser noise up to the $(n-1)$th time-derivative terms of the interspacecraft time delays. Since the proof proceeds similarly for both the unequal-arm Michelson and the Sagnac combinations, in what follows we will just focus on the Michelson combinations.

To take advantage of the dependence of X_3 on its lower-order combinations X_2 and X, let us first focus on the expressions for the residual laser noises in X and X_2. Using our previous notation of Sec. II B, namely, $A_1 \equiv \mathcal{D}_3\mathcal{D}_{3'}$ and $B_1 \equiv \mathcal{D}_{2'}\mathcal{D}_2$, to first order we can write the residual laser noise in X in the following form:

$$\begin{aligned} X = [A_1, B_1]C_1(t) &= C_1(t - L_{A_1}(t) - L_{B_1}(t - L_{A_1}(t))) \\ &\quad - C_1(t - L_{B_1}(t) - L_{A_1}(t - L_{B_1}(t))), \\ &\simeq \dot{C}_1(t - L_{B_1}(t) - L_{A_1}(t))(\dot{L}_{B_1}L_{A_1} - \dot{L}_{A_1}L_{B_1}), \end{aligned} \tag{3.1}$$

where L_{B_1}, L_{A_1} are the two round-trip light times in the two unequal arms and the overdot symbol represents the usual operation of time derivative. Equation (3.1) simply states that the residual laser noise in X is linear in the interspacecraft velocities through an "angular momentum-like" expression. We note that A_1 and B_1 also represent time delays and are time-delay operators in their own right, and therefore follow the same algebraic rules as the elementary delay operators $\mathcal{D}_j$. For reasons that will become clearer later on, we will denote such an expression as

$$S^{(1)} \equiv \dot{L}_{B_1}L_{A_1} - \dot{L}_{A_1}L_{B_1}. \tag{3.2}$$

Since $S^{(1)}$ contains terms linear in velocities, the laser noise in X is not canceled at this order.

Let us now see how we can cancel the terms linear in velocities. Let us consider the following two delay operators: $A_2 \equiv \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2 = A_1B_1$, $B_2 \equiv \mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'} = B_1A_1$. We can formally write the expression of the first-order residual laser noise in X_2 in the following way:

$$\begin{aligned} X_2 = [A_2, B_2]C_1(t) &= C_1(t - L_{A_2}(t) - L_{B_2}(t - L_{A_2}(t))) \\ &\quad - C_1(t - L_{B_2}(t) - L_{A_2}(t - L_{B_2}(t))), \\ &\simeq \dot{C}_1(t - L_{A_2}(t) - L_{B_2}(t))(\dot{L}_{B_2}L_{A_2} - \dot{L}_{A_2}L_{B_2}), \end{aligned} \tag{3.3}$$

where we have denoted with (L_{A_2}, L_{B_2}) the two delays resulting from applying to the laser noise the two operators ($A_2 = \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2$, $B_2 = \mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}$) respectively.

In analogy with the expression of $S^{(1)}$ in Eq. (3.2), which quantifies the first-order expression of the residual laser noise in X, it is convenient to introduce the following combination that defines the magnitude of the first-order residual laser noise in X_2:

$$S^{(2)} \equiv \dot{L}_{B_2}L_{A_2} - \dot{L}_{A_2}L_{B_2}. \tag{3.4}$$

To assess its magnitude we need to expand the two delays (L_{A_2}, L_{B_2}) in terms of the round-trip light times and their time derivatives through the following expressions:

$$\begin{aligned} L_{A_2} &= L_{A_1}(t) + L_{B_1}(t - L_{A_1}(t)) \simeq L_{A_1}(t) + L_{B_1}(t) - \dot{L}_{B_1}(t)L_{A_1}(t), \\ \dot{L}_{A_2} &= \dot{L}_{A_1}(t) + \frac{d}{dt}L_{B_1}(t - L_{A_1}(t)) \simeq \dot{L}_{A_1}(t) + \dot{L}_{B_1}(t) - \frac{d}{dt}(\dot{L}_{B_1}(t)L_{A_1}(t)), \\ L_{B_2} &= L_{B_1}(t) + L_{A_1}(t - L_{B_1}(t)) \simeq L_{B_1}(t) + L_{A_1}(t) - \dot{L}_{A_1}(t)L_{B_1}(t), \\ \dot{L}_{B_2} &= \dot{L}_{B_1}(t) + \frac{d}{dt}L_{A_1}(t - L_{B_1}(t)) \simeq \dot{L}_{B_1}(t) + \dot{L}_{A_1}(t) - \frac{d}{dt}(\dot{L}_{A_1}(t)L_{B_1}(t)). \end{aligned} \tag{3.5}$$

By substituting the expressions given by Eq. (3.5) into Eq. (3.4), after some algebra we get,

$$S^{(2)} = \left(\frac{d}{dt}(\dot{L}_{A_1}L_{B_1}) - (\dot{L}_{A_1} + \dot{L}_{B_1})\right) S^{(1)} + (L_{A_1} + L_{B_1} - \dot{L}_{A_1}L_{B_1})\dot{S}^{(1)}. \tag{3.6}$$

Since $S^{(1)}$ is linear in the interspacecraft velocities, from the above expression we conclude that $S^{(2)}$ (and therefore the residual laser noise in X_2) only contains terms that are quadratic in the relative velocities and linear in the accelerations. The rest of the terms are of higher order than those that we retain here. This is the consequence of the symmetry of the terms which occur in the Taylor expansion and the antisymmetry of the commutator. We provide a detailed mathematical analysis in Appendix which confirms this assertion. Mathematically this is because of the dependence of X_2 on X as shown in Eq. (2.14). Thus we find that the terms linear in velocities are canceled in X_2.

Let us now move on to X_3. From the expression of its residual laser noise given in Eq. (2.17), after defining the two delay operators $A_3 \equiv A_2B_2 = \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}$, $B_3 \equiv B_2A_2 = \mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2$, we can write the expression of its first-order residual laser noise in the following way:

$$X_3 \simeq \dot{C}_1(t - L_{B_3}(t) - L_{A_3}(t))(\dot{L}_{B_3}L_{A_3} - \dot{L}_{A_3}L_{B_3}). \tag{3.7}$$

By defining $S^{(3)}$ to be equal to

$$S^{(3)} \equiv \dot{L}_{B_3}L_{A_3} - \dot{L}_{A_3}L_{B_3}, \tag{3.8}$$

we will now show that $S^{(3)}$ can be written as a linear combination of $S^{(2)}$ and $\dot{S}^{(2)}$, similarly to $S^{(2)}$ being a linear combination of $S^{(1)}$ and $\dot{S}^{(1)}$. To prove this result, we expand the two delays (L_{A_3}, L_{B_3}) and their time derivatives in terms of the delays (L_{A_2}, L_{B_2}) and their time derivatives (which define $S^{(2)}$). We obtain

$$\begin{aligned}
L_{A_3} &= L_{A_2}(t) + L_{B_2}(t - L_{A_2}(t)) \simeq L_{A_2}(t) + L_{B_2}(t) \\
&\quad - \dot{L}_{B_2}(t) L_{A_2}(t), \\
\dot{L}_{A_3} &= \dot{L}_{A_2}(t) + \frac{d}{dt} L_{B_2}(t - L_{A_2}(t)) \simeq \dot{L}_{A_2}(t) + \dot{L}_{B_2}(t) \\
&\quad - \frac{d}{dt}(\dot{L}_{B_2}(t) L_{A_2}(t)), \\
L_{B_3} &= L_{B_2}(t) + L_{A_2}(t - L_{B_2}(t)) \simeq L_{B_2}(t) + L_{A_2}(t) \\
&\quad - \dot{L}_{A_2}(t) L_{B_2}(t), \\
\dot{L}_{B_3} &= \dot{L}_{B_2}(t) + \frac{d}{dt} L_{A_2}(t - L_{B_2}(t)) \simeq \dot{L}_{B_2}(t) + \dot{L}_{A_2}(t) \\
&\quad - \frac{d}{dt}(\dot{L}_{A_2}(t) L_{B_2}(t)).
\end{aligned} \tag{3.9}$$

After substituting Eq. (3.9) into Eq. (3.8), we finally obtain the following expression for $S^{(3)}$ in terms of $S^{(2)}$ and $\dot{S}^{(2)}$:

$$\begin{aligned}
S^{(3)} &= \left(\frac{d}{dt}(\dot{L}_{A_2} L_{B_2}) - (\dot{L}_{A_2} + \dot{L}_{B_2})\right) S^{(2)} \\
&\quad + (L_{A_2} + L_{B_2} - \dot{L}_{A_2} L_{B_2}) \dot{S}^{(2)}.
\end{aligned} \tag{3.10}$$

Since $S^{(2)}$ only contains terms that are either proportional to the square of the interspacecraft velocities or to their relative accelerations, and $\dot{S}^{(2)}$ is further suppressed over $S^{(2)}$ by a time derivative of these terms, from the structure of Eq. (3.10) we conclude that $S^{(3)}$ is of order V smaller than $S^{(2)}$, with V being a typical interspacecraft velocity. Therefore in X_3, terms quadratic in velocities and linear in acceleration are canceled out.

From the derivations of the expressions for $S^{(2)}$ and $S^{(3)}$ above it is now clear that the combination $S^{(4)}$, associated with the residual laser noise in X_4, will cancel laser noise terms that are cubic in the velocity or of order velocity times acceleration or linear in the time derivative of the acceleration, and that in general the expression $S^{(n)}$ associated with the residual laser noise in X_n will depend on the order $n-1$ combinations $S^{(n-1)}$ and $\dot{S}^{(n-1)}$ through a linear relationship similar to those shown by Eqs. (3.6) and (3.10). This is because of the mathematical structure of $S^{(n)}$ and because its defining delays can be written in terms of the delays entering the expression of $S^{(n-1)}$. By induction we therefore conclude that the residual laser noise in the n-order unequal-arms Michelson combination X_n will cancel exactly the laser noise up to $(n-1)$th time derivatives of the interspacecraft time delays.

IV. CONCLUSIONS

Since the exact cancellation of the laser noise for any arbitrary time-dependent delays cannot be achieved [8], in this article we have presented a technique for constructing TDI combinations that cancel the laser noise up to nth-order time-derivative terms of the interspacecraft light-travel times. The lifting procedure, which provides a way for constructing such TDI combinations, entails making two synthesized laser beams going around the array along clockwise and counterclockwise paths a number of times before interfering back at the transmitting spacecraft. In so doing the time variations of the light-travel times is averaged out more and more accurately with the number of loops performed by the beams. We derived the expressions of the third-order TDI combinations (α_3, β_3, γ_3, X_3) as an example application of the lifting procedure, and showed their expressions cancel the laser noise up to terms quadratic in the velocity and linear in the acceleration thanks to the theorem we proved in Sec. III. This result had previously been noticed through a numerical analysis [9] and here we have proved it analytically.

Although the higher-order TDI combinations have been derived using analytic techniques, they could have also been formulated using matrices. This would have resulted in the same higher-order TDI observables derived here albeit numerically [10,13,14]. This implies that representations of operators using matrices lend themselves to easy numerical manipulations.

It is important to note that currently planned GW missions do not need to cancel laser noise terms quadratic in the velocities or linear in the accelerations because of their benign interspacecraft relative velocities (≈10 m/s) [1–5]. However, future interplanetary missions capable of measuring interspacecraft relative Doppler of 10 km/s or larger will need to synthesize third-order TDI combinations to suppress the laser noise to the required levels.

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APPENDIX: PROOF OF THE ASSERTION

For completeness we start with the expression of the Taylor expansion of two delays A_1 and B_1 applied successively [see Eq. (2.6)],

$$\begin{aligned}
(A_1 B_1) C &= C(t - L_{A_1} - L_{B_1}) + \left[L_{A_1} v_{B_1} - \frac{1}{2} L_{A_1}^2 f_{B_1}\right] \dot{C} \\
&\quad + \frac{1}{2} L_{A_1}^2 v_{B_1}^2 \ddot{C},
\end{aligned} \tag{A1}$$

where we have retained terms quadratic in v and linear in f and where $v_{A_1} = \dot{L}_{A_1}$ and $f_{A_1} = \ddot{L}_{A_!}$ etc. Also instead of writing the $\mathcal{D}$ operators explicitly, in order to avoid clutter, we have indicated those operators in terms of their delays. Further we mention that the time derivatives are evaluated at the delayed times.

Interchanging A_1 and B_1 provides the action B_1A_1 on C. Then combining the two expressions we obtain the commutator:

$$[A_1, B_1]C = \left[(L_{A_1}v_{B_1} - L_{B_1}v_{A_1}) - \frac{1}{2}(L_{A_1}^2 f_{B_1} - L_{B_1}^2 f_{A_1})\right]\dot{C} + \frac{1}{2}[L_{A_1}^2 v_{B_1}^2 - L_{B_1}^2 v_{A_1}^2]\ddot{C}. \tag{A2}$$

Now we proceed to the second nested commutator of the delays $A_2 = A_1B_1$ and $B_2 = B_1A_1$. We note that the commutator of A_2 and B_2 gives formally the same expression as Eq. (A2) with 1 replaced by 2. We write this commutator as

$$[A_2, B_2]C = \left[U - \frac{1}{2}V\right]\dot{C} + \frac{1}{2}W\ddot{C}, \tag{A3}$$

where $U = L_{A_2}v_{B_2} - L_{B_2}v_{A_2}$, $V = L_{A_2}^2 f_{B_2} - L_{B_2}^2 f_{A_2}$, and $W = L_{A_2}^2 v_{B_2}^2 - L_{B_2}^2 v_{A_2}^2$. We now need to write U, V, and W and the commutator in terms of A_1 and B_1. For this, we require the following relations:

$$\begin{aligned} L_{A_2} &= L_{A_1} + L_{B_1} - L_{A_1}v_{B_1} + \frac{1}{2}L_{A_1}^2 f_{B_1}, \\ L_{B_2} &= L_{B_1} + L_{A_1} - L_{B_1}v_{A_1} + \frac{1}{2}L_{B_1}^2 f_{A_1}, \\ v_{A_2} &= v_{A_1} + v_{B_1} - v_{A_1}v_{B_1} - L_{A_1}f_{B_1}, \\ v_{B_2} &= v_{B_1} + v_{A_1} - v_{B_1}v_{A_1} - L_{B_1}f_{A_1}. \end{aligned} \tag{A4}$$

Note that we write all expressions so that a given quantity is expressed up to the required order only. We first compute U. Because of the symmetry of the terms and the antisymmetry of the commutator, several terms cancel out. Further we drop terms of order $o(v^3)$, $o(vf)$, $o(f^2)$. The result is

$$U = (v_{A_1} + v_{B_1})(L_{B_1}v_{A_1} - L_{A_1}v_{B_1}) + (L_{A_1} + L_{B_1})(L_{A_1}f_{B_1} - L_{B_1}f_{A_1}) \equiv -S^{(2)}. \tag{A5}$$

From here we see that $U = -S^{(2)}$ is quadratic in velocities and linear in acceleration. We now show that this is the only term that contributes at this order.

We now show that V and W are of higher order and so are zero at this order. Differentiating the velocity equations in Eq. (A4) with respect to time we obtain to this order [the rest of the terms are $o(vf)$ and $o(\dot{f})$],

$$f_{A_2} = f_{A_1} + f_{B_1} = f_{B_2}. \tag{A6}$$

A similar argument shows that $L_{A_2} = L_{A_1} + L_{B_1} = L_{B_2}$ (only these terms contribute at the required order). Thus $V = 0$ at this order.

We argue similarly for W. We have $L_{A_2}^2 v_{B_2}^2 = (L_{A_1} + L_{B_1})^2(v_{A_1} + v_{B_1})^2 = L_{B_2}^2 v_{A_2}^2$ at this order. Thus $W = 0$. So it is only the *leading* term U that contributes at this order which is essentially $S^{(2)}$.

One can further ascertain that $S^{(3)} = 0$ at this order. To obtain $S^{(3)}$ (except for a sign) we replace 1 by 2 in Eq. (A5). Thus,

$$S^{(3)} = (v_{A_2} + v_{B_2})(L_{B_2}v_{A_2} - L_{A_2}v_{B_2}) + (L_{A_2} + L_{B_2})(L_{A_2}f_{B_2} - L_{B_2}f_{A_2}). \tag{A7}$$

At this order, apart from $L_{A_2} = L_{B_2}$ and $f_{A_2} = f_{B_2}$ as shown above, we also have $v_{A_2} = v_{A_1} + v_{B_1} = v_{B_2}$. This produces the required result that $S^{(3)} \approx 0$.

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