Second-generation time-delay interferometry

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Time-delay interferometry (TDI) is the data processing technique that cancels the large laser phase fluctuations affecting the heterodyne Doppler measurements made by unequal-arm space-based gravitational wave interferometers. The space of all TDI combinations was first derived under the simplifying assumption of a stationary array, for which the three time-delay operators commute. In this model, any element of the TDI space can be written as a linear combination of four TDI variables, the generators of the "first-generation" TDI space. To adequately suppress the laser phase fluctuations in a realistic array configuration, the rotation of the array and the time dependence of the six interspacecraft light travel times has to be accounted for. In the case of the Laser Interferometer Space Antenna (LISA), a European Space Agency mission characterized by slowly time varying armlengths, it has been possible to identify data combinations that, to first order in the interspacecraft velocities, either exactly cancel or suppress the laser phase fluctuations below the level identified by the noise sources intrinsic to the heterodyne measurements (the so-called "secondary" noises). Here we reanalyze the problem of *exactly* canceling the residual laser noise terms linear in the interspacecraft velocities. We find that the procedure for obtaining elements of the second-generation TDI space can be generalized in an iterative way. This allows us to "lift up" the generators of the first-generation TDI space and construct elements of the higher order TDI space whose gravitational wave sensitivities are equal to those of their first-generation counterparts.

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I. INTRODUCTION

The Laser Interferometer Space Antenna (LISA) is a space mission proposed by the European Space Agency to observe gravitational waves (GW) in the millihertz frequency band. LISA will rely on an array of three identical spacecraft that exchange coherent laser beams along the three 2.5-million-kilometer arms of the resulting giant (almost) equilateral triangle. The heliocentric trajectories of the three spacecraft result in armlengths that are unequal and weakly time dependent with interspacecraft relative velocities $\lesssim 10\ \mathrm{m/s}$. Since these velocities are negligible compared to the speed of light, we are justified in retaining only first order terms in the velocities in our considerations. The frequency noise of the LISA stabilized lasers dominates the other secondary noises by 7 or more orders of magnitude and must be removed or sufficiently suppressed to achieve the requisite sensitivity. By linearly combining the appropriately delayed six one-way interspacecraft Doppler measurements, we can construct data combinations—the time-delay interferometry (TDI) combinations—that cancel (or sufficiently suppress) the laser frequency noise while retaining sensitivity to GWs.

The simplest assumption is to regard the armlengths to be constant and consider only three time delays. This means that the light travel time between spacecraft i to j is the same as between j to i. This is not true in general because the LISA triangle rotates once in a year. The Sagnac effect implies that the up and down optical paths are unequal. The TDI space that arises from the assumption of three constant armlengths is the so-called *first-generation* TDI [1–3]. A rigorous mathematical foundation for this case was laid in [4] proving that the TDI space was a linear structure called in the literature as the *first module of*

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syzygies [5,6] which is a module over the polynomial ring of the three time-delay operators. A neat solution was possible because the delay operators commute and form a commutative polynomial ring. Hilbert's theorem guarantees that in a commutative polynomial ring over a field, all ideals are finitely generated or the ring is Noetherian. This implies that the Gröbner basis algorithm terminates and finally leads to a finite set of generators for the module. This module is a *kernel* of a homomorphism [7] or the TDI maps the laser noise to zero and therefore forms a null space. It has been shown that the module is generated by a set of four generators, the simplest and most useful set being α, β, γ, and ζ, the Sagnac combinations.

The next level of simplification is to consider the Sagnac effect so that now we have six time delays but they are considered to be time independent. This case can also be solved exactly [8,9] and results in six generators for the first module of syzygies. These form the so-called 1.5-generation TDI space.

The most general case consists of TDI combinations where the array is rotating and the six time delays are time dependent. In this case the operators do not commute and one ends up with a noncommutative polynomial ring, whose elements are strings of operators or "words" as they are called in the literature. In the past, one of the authors (S. V. D.) has attempted to compute the analogous Gröbner basis for the noncommutative case but found that the algorithm did not terminate. Others have tried to use *Mathematica* towards the same goal but have not succeeded. Therefore, it seems that the Gröbner basis is infinite and this approach seems to be intractable. In the case of LISA, however, the armlengths are slowly changing in time and the problem therefore can be treated like a "perturbation" over the static case and the results obtained thereby suitably generalized.

In this paper, we will first study the TDI space with six different delays that are slowly time varying—we will consider terms only to first order in the interspacecraft relative velocities. In the past this case has been considered [7,8,10,11] with partial solutions for the so-called second-generation TDI space. In recent publications [12,13] an alternative approach was proposed, in which second-generation TDI combinations were obtained through the use of a computer program. Its underlining algorithm relied on geometric TDI [14] and searched for combinations that would suppress the laser noise below the level identified by their secondary noises. Although this approach identified a significantly large number of second-generation TDI combinations, it could not check for their independence nor assess the dimensionality of the second-generation TDI space. An attempt to answer these questions has been presented in [15], where the new TDI channels derived in [12] were related to the Sagnac generators $\alpha, \beta, \gamma, \zeta$ of the first-generation TDI space. Although the established relationship cannot be mathematically exact, it is nevertheless accurate enough for modeling the residual noises of these second-generation TDI expressions. Its drawback, however, is relying on Sagnac observables containing only the three delay operators characteristic of a stationary array.

Finally, an analytic approach has been proposed [16] for finding elements of the second-generation TDI space. It entails a generalization of work presented in [11] for analytically deriving second-generation unequal-arm Michelson combinations. In [16] new Sagnac-like combinations as well as a new set of expressions for the monitor, beacon, and relay [8] have been presented.

In this paper we propose instead a different approach from those cited above for identifying TDI combinations that cancel exactly the laser noise when the delays are characterized by small interspacecraft velocities. We do so by also using only analytic techniques. Recently matrix methods have also been employed, which lead to TDI observables albeit numerically [12,17–19]. Although the TDI combinations we will derive in this article can be recast in matrix form, we will not do that here. In our approach we first rewrite the elements of a basis of the first-generation TDI space in terms of the six delay operators. Then we show that their corresponding second-generation TDI expressions can be obtained by acting on specific combinations of the data entering their expressions with uniquely identified polynomials of the six delays. This so-called "lifting" operation is key to our method as it allows us to generalize the main property of a basis of the first-generation TDI space: elements of the second-generation TDI space can be obtained by taking linear combinations of properly delayed lifted basis. In physical terms, the operation of lifting corresponds to two light beams each propagating clockwise and counterclockwise several times around the array before being made to interfere onboard the transmitting spacecraft. In so doing the time variations and the Sagnac effect on the light travel times get averaged out [8].

The paper is organized as follows. In Sec. II we review some of the past relevant results, which will be required here, by deriving a suitable set of four elements of the first-generation TDI space that can *uniquely* be written in terms of the six time-dependent delays and generate this space in the limit of a stationary array. Although in the stationary configuration the basis usually adopted included the four Sagnac combinations $\alpha, \beta, \gamma, \zeta, \zeta$ loses its uniqueness when trying to incorporate the six time-dependent delays in its definition. Also it cannot be interpreted as the result of the interference of two beams that have been propagating along two different paths and a straightforward geometric interpretation to ζ is lacking [8,16]. To avoid this complication, we use instead the four data combinations α, β, γ, X, as generators of the first-generation TDI space, with X being the usual unequal-arm Michelson combination. This is possible because ζ is linearly related to α, β, γ, X [2].

After deriving the expressions for the residual laser noises in specific data combinations entering the expressions of α, β, γ, X, in Sec. III we present useful identities of the delay operators valid with six, time-varying delays characterized by small interspacecraft velocities. These identities are used to derive the second-generation TDI combinations that cancel the laser noise up to the velocities of the interspacecraft light travel times. We call this technique "lifting" as it allows us to derive the corresponding elements in the second-generation TDI space by starting with the basis elements of the first generation. By then suitably delaying and linearly combining the lifted basis of the first-generation TDI space, one can generate elements of the higher-order space. As an application we derive expressions of (i) ζ-like combinations that exactly cancel the laser noise while suppressing (like ζ) the gravitational wave signal in the low part of the accessible frequency band and (ii) second-generation TDIs containing only four-link measurements (i.e. the beacon P_2, monitor E_2, and relay U_2 combinations). In Sec. VI we finally present our comments on our findings and our conclusions.

II. THE FIRST-GENERATION TDI SPACE

Here we present a brief summary of the derivation of the TDI space valid for a stationary array. We start by writing the one-way Doppler data $y_i, y_{i'}$ in terms of the laser noises using the notation introduced in [7,19]. We index the one-way Doppler data as follows: the beam arriving at spacecraft i has subscript i and is primed or unprimed depending on whether the beam is traveling clockwise or counterclockwise around the interferometer array, with the sense defined by a chosen orientation of the array (see Fig. 1). We define the delay operators $\mathcal{D}_i$ by $\mathcal{D}_i y(t) = y(t - L_i)$ where L_i is the travel time spent by the light to travel the ith arm (the speed of light has been assumed to be equal to 1). The assumption of a stationary array implies the following expressions for the six one-way interspacecraft Doppler measurements[1]:

$$\begin{aligned} y_1 &= \mathcal{D}_3 C_2 - C_1, & y_{1'} &= \mathcal{D}_2 C_3 - C_1, \\ y_2 &= \mathcal{D}_1 C_3 - C_2, & y_{2'} &= \mathcal{D}_3 C_1 - C_2, \\ y_3 &= \mathcal{D}_2 C_1 - C_3, & y_{3'} &= \mathcal{D}_1 C_2 - C_3. \end{aligned} \tag{2.1}$$

The problem of identifying all possible TDI combinations associated with the six one-way Doppler measurements becomes one of determining six polynomials, $q_i, q_{i'}$, in the delay operators $\mathcal{D}_i, i = 1, 2, 3$. The polynomials $q_i, q_{i'}$ satisfy the equation $\sum_{i=1}^{3} q_i . y_i + \sum_{i'=1}^{3} q_{i'} . y_{i'} = 0$, where the equality means "zero laser noises."

[1]Besides the primary interspacecraft Doppler measurement $y_i, y_{i'}$ that contains the gravitational wave signal, other metrology measurements are made onboard the LISA spacecraft. This is because each spacecraft is equipped with two lasers and two proof masses of the onboard drag-free subsystem. It has been shown [7], however, that these onboard measurements can be properly delayed and linearly combined with the interspacecraft measurements to make the realistic LISA interferometry configuration equivalent to that of a system with only three lasers and six one-way interspacecraft measurements.

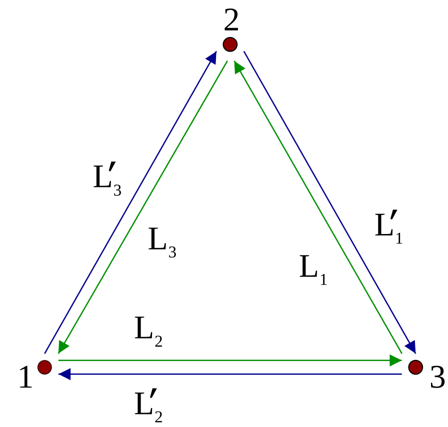

FIG. 1. Schematic array configuration. The spacecraft are labeled 1, 2, and 3, and the optical paths are denoted by L_i, L'_i with the index i corresponding to the opposite spacecraft.

It can be shown that the resulting TDI space is the first module of syzygies [2,4,7]. We will be mainly concerned with the Sagnac TDI observables $\alpha, \beta, \gamma, \zeta$ that generate the TDI space [2,4] because these observables generate the module. We will also consider the Michelson TDI X because of its inherent simplicity, which will act as a guide for the other cases. We therefore list these TDI generators below and write them as six tuple polynomial vectors $(q_i, q_{i'})$ (in this notation the data streams $y_i, y_{i'}$ are implicit):

$$\begin{aligned} \alpha &= (1, \mathcal{D}_3, \mathcal{D}_3\mathcal{D}_1, -1, -\mathcal{D}_2\mathcal{D}_1, -\mathcal{D}_2), \\ \beta &= (\mathcal{D}_1\mathcal{D}_2, 1, \mathcal{D}_1, -\mathcal{D}_3, -1, -\mathcal{D}_3\mathcal{D}_2), \\ \gamma &= (\mathcal{D}_2, \mathcal{D}_2\mathcal{D}_3, 1, -\mathcal{D}_1\mathcal{D}_3, -\mathcal{D}_1, -1), \\ \zeta &= (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, -\mathcal{D}_1, -\mathcal{D}_2, -\mathcal{D}_3). \end{aligned} \tag{2.2}$$

The observables α, β, γ, ζ perfectly cancel the laser frequency noise when the armlengths are time independent. In this paper we propose to go beyond this simple case, where the armlengths weakly depend on time. Our goal is to generalize the first-generation TDI space to the situation in which the armlengths vary slowly.

We will find that α and its cyclic permutations β and γ can be converted into second-generation TDI with the help of commutators and some algebraic manipulation. But the TDI ζ is not so straightforward as it cannot be thought of as the result of the interference of two beams propagating along two different paths. However, we may switch to another set of generators, namely, α, β, γ and the unequal-arm Michelson combination X. This is possible because of the following relationship [2] between $\zeta$and α, β, γ, X:

$$\zeta = \mathcal{D}_1 X - \mathcal{D}_2\mathcal{D}_3\alpha + \mathcal{D}_2\beta + \mathcal{D}_3\gamma, \tag{2.3}$$

where X is

$$X = (1 - \mathcal{D}_2^2, 0, (\mathcal{D}_3^2 - 1)\mathcal{D}_2, \mathcal{D}_3^2 - 1, (1 - \mathcal{D}_2^2)\mathcal{D}_3, 0). \tag{2.4}$$

Equation (2.3) means that any linear combination of the generators α, β, γ, ζ is also a linear combination of α, β, γ, X. This implies that α, β, γ, and X is another generating set for the module of syzygies. We will therefore include the derivation of the second-generation combination X_2 that had already been derived in earlier publications [8,10].

At the zeroth order in the interspacecraft velocity the laser noise cancels out for the first-generation TDI, those given in Eq. (2.2) and also the Michelson X. But at the next order in the velocities, the laser noise does not cancel out completely in these TDIs, making it larger than their remaining noises. This we call residual laser noise and denote the corresponding TDI by the subscript *res*:

$$\begin{aligned}
\alpha_{\mathrm{res}} &= (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_3)C_1,\\
\beta_{\mathrm{res}} &= (\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3 - \mathcal{D}_3\mathcal{D}_2\mathcal{D}_1)C_2,\\
\gamma_{\mathrm{res}} &= (\mathcal{D}_2\mathcal{D}_3\mathcal{D}_1 - \mathcal{D}_1\mathcal{D}_3\mathcal{D}_2)C_3,\\
X_{\mathrm{res}} &= (\mathcal{D}_3\mathcal{D}_3\mathcal{D}_2\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_2\mathcal{D}_3\mathcal{D}_3)C_1.
\end{aligned} \tag{2.5}$$

Since the above expressions contain products of operators which are permutations of each other and occur with opposite sign, at zeroth order the laser noise cancels out but at first order the velocity terms [as we will see in Eq. (3.4) below] multiplying the $\dot{C}$ term do not cancel out. These residual laser noises must be canceled to achieve the requisite sensitivity.

III. TDI WITH SIX TIME-DEPENDENT TIME DELAYS

A. The general model of LISA

We started with the first-generation TDI because we can cleanly derive the exact generators that completely span the TDI space. Our idea is to use these foundational results to generalize to the realistic model of LISA. We will achieve this by what we call the lifting procedure. This procedure is described in Sec. IV. We now set up the analysis for six time-dependent time delays. Because of the Sagnac effect due to the rotation of the LISA constellation, the armlength from say spacecraft i to j is not the same as the one from j to i. Therefore $L_i \neq L'_i$ and so we have six unequal time-dependent armlengths. The corresponding operators are now labeled as $\mathcal{D}_i$ and $\mathcal{D}_{i'}$.

The one-way phase measurements therefore assume the following forms:

$$\begin{aligned}
y_1 &= \mathcal{D}_3 C_2 - C_1, & y_{1'} &= \mathcal{D}_{2'} C_3 - C_1,\\
y_2 &= \mathcal{D}_1 C_3 - C_2, & y_{2'} &= \mathcal{D}_{3'} C_1 - C_2,\\
y_3 &= \mathcal{D}_2 C_1 - C_3, & y_{3'} &= \mathcal{D}_{1'} C_2 - C_3,
\end{aligned} \tag{3.1}$$

where we have adopted the labeling convention shown in Fig. 1. In it the phase difference data to be analyzed is indexed as follows: the beam arriving at spacecraft i has subscript i and is primed or unprimed depending on whether the beam is traveling clockwise or counterclockwise (the sense defined here with reference to Fig. 1) around the array's triangle, respectively. Thus, as seen in the figure, y_1 is the phase difference time series measured at reception at spacecraft 1 with transmission from spacecraft 2 (along L_3). The polynomials q_i, q'_i satisfy somewhat more general equations [7].

B. Slowly time-varying armlengths and vanishing commutators

If the armlengths are time dependent, then the operators do not commute and the laser noise will not cancel. However, if the armlengths are slowly varying we can make a Taylor expansion of the operators and keep terms only to first order in $\dot{L}_i$ and $\dot{L}'_i$ or linear in velocities.

Let us consider the effect of n operators $\mathcal{D}_{k_1}, \ldots, \mathcal{D}_{k_n}$ applied on the laser noise $C(t)$. The operators could refer to either L_i or $L_{i'}$. We do not write the primes explicitly in order to avoid clutter but the identities that we derive hold in either case. Instead of writing $\mathcal{D}_{k_p}$ we may denote the same by just k_p where k_p take any of the values 1, 2, 3, 1′, 2′, 3′. Then as shown in [7,20] we have

$$k_n k_{n-1} \ldots k_2 k_1 C(t) = C\left[t - \sum_{p=1}^{n} L_{k_p}\right] + \left[\sum_{j=2}^{n} L_{k_j} \sum_{m=1}^{j-1} \dot{L}_{k_m}\right] \dot{C}\left[t - \sum_{p=1}^{n} L_{k_p}\right]. \tag{3.2}$$

Let us interpret the right-hand side of this equation. The first term is just the laser noise at a delayed time that is equal to the sum of the delays at time t. If the armlengths were constant this would be the only term that would be present and the operators would commute leading to first-generation TDI. Note that the second term *multiplies* the $\dot{C}$ evaluated at the delayed time. This term makes the operators noncommutative. But the noncommutativity is small because the armlengths are slowly varying i.e. $\dot{L} \ll 1$—it is linear in the velocities. The first term is of zeroth order in velocities. Since here we are only concerned with the second term, we will only write the second term assuming that the zeroth order term has been canceled exactly in the expressions. Further, in order to avoid clutter, we will not write C or $\dot{C}$ when there is no cause for confusion. We may also write $v_{k_p} = \dot{L}_{k_p}$. Then with this understanding we may write Eq. (3.2) as

$$k_n k_{n-1} \ldots k_2 k_1 = \sum_{j=2}^{n} L_{k_j} \sum_{m=1}^{j-1} v_{k_m}. \tag{3.3}$$

Note that the k_p need not be distinct—the operators may repeat. We write the first few products explicitly as

$$\begin{aligned}\mathcal{D}_2\mathcal{D}_1 &= L_2 v_1,\\ \mathcal{D}_3\mathcal{D}_2\mathcal{D}_1 &= L_2 v_1 + L_3(v_1+v_2),\\ \mathcal{D}_4\mathcal{D}_3\mathcal{D}_2\mathcal{D}_1 &= L_2 v_1 + L_3(v_1+v_2)\\ &\quad + L_4(v_1+v_2+v_3).\end{aligned} \tag{3.4}$$

It was shown in [7,11] that certain commutators cancel the laser noise under the approximation we are making. Let $x_1, x_2, \ldots, x_n$ and $z_1, z_2, \ldots, z_n$ be delay operators. Then it follows from Eq. (3.3) that

$$[x_1 x_2 \ldots x_n, z_1 z_2 \ldots z_n] = \sum_{k=1}^{n} L_{x_k} \sum_{m=1}^{n} v_{z_m} - \sum_{m=1}^{n} L_{x_m} \sum_{k=1}^{n} v_{z_k}. \tag{3.5}$$

Let σ be a permutation on n symbols. Then $x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}$ is a permutation of $x_1, x_2, \ldots, x_n$, then it is easy to show that

$$[x_1 x_2 \ldots x_n, x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}] = 0. \tag{3.6}$$

It was shown in [11] that a large number of Michelson type TDI can be generated by relying on Eq. (3.6) and, more recently [16], those results have been generalized to find many other elements of the second-generation TDI space such as the Sagnac, symmetric Sagnac, monitor, beacon, and relay.

IV. THE LIFTING PROCEDURE

We first need to derive the expressions of the four generators, α, β, γ, X, of the first-generation TDI formulation that include the six delays i, i' $i, i' = 1, 2, 3, 1', 2', 3'$. Since these combinations correspond to beams propagating clockwise and counterclockwise, we can then generalize the procedure for identifying combinations that suppress the laser noise to the required levels [8,10]. This is done by making each beam propagate clockwise and counterclockwise a number of times such that the resulting data combinations exactly cancel the laser noise up to the velocities of the six delays. This procedure, which we now call lifting, is unique and can be applied iteratively an arbitrary number of times. It should be emphasized that some elements of the second-generation TDI space, like the Sagnac combinations α, β, γ, require more than two lifting iterations to exactly cancel the laser noise up to the linear velocity terms [7,8]. Therefore we will refer to the space of the second-generation TDI space as those combinations that *exactly* cancel the laser noise up to the linear velocity terms.

A. The unequal-arm Michelson X

The X combination includes the four one-way Doppler measurements, $(y_1, y_{1'}, y_{2'}, y_3)$ from the two arms centered on spacecraft 1. In what follows we will present the method discussed in [7,8,10] for obtaining the second-generation TDI X_2, and generalize this approach to derive other unequal-arm Michelson combinations. Let us consider the following synthesized two-way Doppler measurements:

$$\begin{aligned} X_\uparrow &\equiv y_1 + \mathcal{D}_3 y_{2'} = (\mathcal{D}_3\mathcal{D}_{3'} - I)C_1,\\ X_\downarrow &\equiv y_{1'} + \mathcal{D}_{2'} y_3 = (\mathcal{D}_{2'}\mathcal{D}_2 - I)C_1.\end{aligned} \tag{4.1}$$

As we know, the first-generation TDI combination X, is equal to the expression

$$\begin{aligned} X &\equiv (\mathcal{D}_3\mathcal{D}_{3'} - I)X_\downarrow - (\mathcal{D}_{2'}\mathcal{D}_2 - I)X_\uparrow\\ &= [\mathcal{D}_3\mathcal{D}_{3'}, \mathcal{D}_{2'}\mathcal{D}_2]C_1.\end{aligned} \tag{4.2}$$

It is easy to see the above commutator is different from zero when the delays are functions of time and, to first order, are in fact proportional to the interspacecraft relative velocities. To derive the second-generation TDI combination X_2, which cancels exactly the laser noise up to linear velocity terms, we rewrite the above expression for X in terms of its two synthesized beams. They are equal to

$$\begin{aligned} X_{\uparrow\uparrow} &\equiv \mathcal{D}_{2'}\mathcal{D}_2 X_\uparrow + X_\downarrow = (\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'} - I)C_1,\\ X_{\downarrow\downarrow} &\equiv \mathcal{D}_3\mathcal{D}_{3'} X_\downarrow + X_\uparrow = (\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2 - I)C_1.\end{aligned} \tag{4.3}$$

The X_2 expression can be derived by repeating the same procedure used for deriving X. This results in the expression

$$\begin{aligned} X_2 &\equiv (\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2 - I)X_{\uparrow\uparrow} - (\mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'} - I)X_{\downarrow\downarrow}\\ &= [\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2, \mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}]C_1 = 0,\end{aligned} \tag{4.4}$$

where the equality to zero means "up to linear velocity terms" and it is a consequence of the general property of the commutators of the delay operators proved in the previous section. It is clear that the iterative procedure we have implemented for deriving both X and X_2 can be repeated to obtain the expression for other unequal-arm Michelson combinations. Lastly we note that, because the magnitudes of the frequency fluctuations associated with a GW signal and the secondary noises in X_2 are significantly smaller than those of a laser, the commutator of two delay operators applied to them results in relative frequency fluctuations that are about 7 orders of magnitude smaller than their values and can therefore be regarded as equal to zero. This means that the order by which two delay operators act on a GW signal and the secondary noises can be ignored. This observation implies that their contributions to X_2, $X_2^{\mathrm{GW,N}}$

are related to those in X, $X^{GW,N}$ through the following relationship:

$$X_2^{\mathrm{GW,N}} = (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)X^{\mathrm{GW,N}}. \quad (4.5)$$

Equation (4.5), which has already appeared in the literature [21], follows from Eqs. (4.4) and (4.2) after some simple algebraic manipulations that account for the commutativity of the delay operators when applied to a GW signal and the secondary noises. It states the GW signal and secondary noises present in X_2 are related to those in X through the operator $(I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)$. It also says that the GW sensitivity of X_2 is equal to that of X because the Fourier transfer function of the operator $(I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)$ multiplies both the GW signal and the noise in X and thus cancels out. In general, if A and B are two TDI observables such that $A = p(\mathcal{D}_i, \mathcal{D}_{i'})B$, where $p(\mathcal{D}_i, \mathcal{D}_{i'})$ is a polynomial in the delay operators $\mathcal{D}_i$ and $\mathcal{D}_{i'}$ then because the same transfer function scales both the signal and the noise in A and B, the sensitivities of A and B are identical.

We will be using Eq. (4.5) later on when deriving other second-generation TDI combinations.

B. The Sagnac combination α

The α combination represents a synthesized Sagnac interferometer. In it two synthesized light beams interfere onboard spacecraft 1 after making a clockwise and counterclockwise loop around the array. From simple geometric considerations on the delays and paths traveled by the two synthesized beams it is easy to derive the following expression for α:

$$\alpha = [y_1 + \mathcal{D}_3 y_2 + \mathcal{D}_3\mathcal{D}_1 y_3] - [y_{1'} + \mathcal{D}_{2'} y_{3'} + \mathcal{D}_{2'}\mathcal{D}_{1'} y_{2'}]. \quad (4.6)$$

After substituting Eqs. (3.1) into Eq. (4.6) we find the expression of the residual laser noise $C_1(t)$ in α to be equal to

$$\alpha_{res} = (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'})C_1. \quad (4.7)$$

The 1.5-generation TDI Sagnac observables were obtained by making each beam go around the array twice, in clockwise and counterclockwise directions. In so doing the effects of rotation could get canceled exactly, while linear terms in the velocities multiplying the laser noise would get adequately suppressed below the secondary noises. As we will show below, exact cancellation of the laser noise up to linear velocity terms can be achieved by having the beams make additional loops around the array.

Let us consider the two beams forming alpha,

$$\alpha_\uparrow \equiv y_1 + D_3 y_2 + D_3 D_1 y_3 = (D_3 D_1 D_2 - I)C_1,$$
$$\alpha_\downarrow \equiv y_{1'} + D_{2'} y_{3'} + D_{2'} D_{1'} y_{2'} = (D_{2'} D_{1'} D_{3'} - I)C_1. \quad (4.8)$$

The 1.5-generation TDI Sagnac observable, $\alpha_{1.5}$, was then obtained by forming the following linear combination of $\alpha_\uparrow$ and $\alpha_\downarrow$:

$$\alpha_{1.5} \equiv (D_{2'} D_{1'} D_{3'} - I)\alpha_\uparrow - (D_3 D_1 D_2 - I)\alpha_\downarrow = [D_{2'} D_{1'} D_{3'}, D_3 D_1 D_2]C_1. \quad (4.9)$$

From the properties of commutators derived in the previous section, we recognize that the right-had side of Eq. (4.9) does not cancel the laser noise terms containing the velocities.[2] However, by applying our iterative procedure one more time this can be achieved. Let us first write the following two expressions, which take into account Eq. (4.9):

$$\alpha_{\uparrow\uparrow} = D_{2'} D_{1'} D_{3'} \alpha_\uparrow + \alpha_\downarrow = (D_{2'} D_{1'} D_{3'} D_3 D_1 D_2 - I)C_1,$$
$$\alpha_{\downarrow\downarrow} = \alpha_\uparrow + D_3 D_1 D_2 \alpha_\downarrow = (D_3 D_1 D_2 D_{2'} D_{1'} D_{3'} - I)C_1. \quad (4.10)$$

From the above equation we then obtain the following expression for α_2:

$$\alpha_2 \equiv (D_3 D_1 D_2 D_{2'} D_{1'} D_{3'} - I)\alpha_{\uparrow\uparrow} - (D_{2'} D_{1'} D_{3'} D_3 D_1 D_2 - I)\alpha_{\downarrow\downarrow} = [D_{2'} D_{1'} D_{3'} D_3 D_1 D_2, D_3 D_1 D_2 D_{2'} D_{1'} D_{3'}]C_1. \quad (4.11)$$

We may notice that the operator that applies to C_1 in Eq. (4.11) is the commutator of two delay operators, each being the product of the same number of primed and unprimed delay operators and related by permutations of their indices. From the commutator identities derived in the previous section, we conclude that such a commutator results in the exact cancellation of the laser noise up to linear velocity terms. The iterative process highlighted above can of course be repeated, resulting in other TDI combinations. Finally we now provide the expression of $\alpha_2^{\mathrm{GW,N}}$ in terms of $\alpha^{\mathrm{GW,N}}$ [21], which follows from Eqs. (4.11), (4.10), (4.9), (4.8):

$$\alpha_2^{\mathrm{GW,N}} = (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)\alpha^{\mathrm{GW,N}}. \quad (4.12)$$

[2]Although the 1.5-generation α combination was also referred to in the literature as being an element of the second-generation TDI space because it suppresses the laser noise below the secondary noises, here we will refer to it as $\alpha_{1.5}$ since it does not exactly cancel the laser noise up to the linear velocity terms.

Equation (4.12) above reflects the fact that the delay operators can be treated as constant and that the inequality between the primed and unprimed delays can also be disregarded when acting on a GW signal and the secondary noises in α_2. Like in the case of X_2 and X, here too α_2 and α have the same sensitivity to gravitational waves as the same Fourier transfer function multiplies both the GW signal and the secondary noises of α.

As it will be shown below, Eqs. (4.5) and (4.12) will play a key role in the derivation of other second-generation TDI combinations by properly delaying and linearly combining the four observables (α_2, β_2, γ_2, X_2).

V. THE SECOND-GENERATION TDI SPACE

In what follows we first show that any element of the first-generation TDI space can be lifted up to the second-generation space. We emphasize, however, that lifting is not a bijection between the two TDI spaces. There exist in fact an infinite number of lifted combinations showing the same sensitivity to GWs as their first-generation counterpart. As an application of this general theorem, we then derive the expressions for the symmetric Sagnac combination ζ_2, the monitor E_2, the beacon P_2, and the relay U_2. This is done by taking specific combinations of the lifted basis (α_2, β_2, γ_2, X_2). Although there already exist expressions in the literature for the E_2, P_2, U_2 combinations that cancel the laser noise up to linear velocity terms [7,8], the $\zeta_{1.5}$ combinations [8] only suppress the laser noise below the secondary noises. For this reason we have included the derivation of ζ_2, which cancels exactly the laser noise terms linear in the velocity.

A. Lifting the first-generation TDI space

Let us consider the following arbitrary element of the first-generation TDI space:

$$\rho \equiv \lambda_\alpha \alpha + \lambda_\beta \beta + \lambda_\gamma \gamma + \lambda_X X, \tag{5.1}$$

where $(\lambda_\alpha, \lambda_\beta, \lambda_\gamma, \lambda_X)$ are arbitrary polynomials of the six delays. Let us also take the following arbitrary linear combination of $(\alpha_2, \beta_2, \gamma_2, X_2)$ [the lifted counterparts of $(\alpha, \beta, \gamma, X)$]:

$$\rho_2 \equiv \lambda_{\alpha_2} \alpha_2 + \lambda_{\beta_2} \beta_2 + \lambda_{\gamma_2} \gamma_2 + \lambda_{X_2} X_2, \tag{5.2}$$

where $(\lambda_{\alpha_2}, \lambda_{\beta_2}, \lambda_{\gamma_2}, \lambda_{X_2})$ are also arbitrary polynomials of the six delays. Since $(\alpha_2, \beta_2, \gamma_2, X_2)$ cancel exactly the laser noises, it is clear that any linear combination of them [such as that given by Eq. (5.2)] is also laser noise-free. Since (α_2, β_2, γ_2, X_2) only contain the GW signal and the secondary noises, we can replace in Eq. (5.2) their expressions in terms of the first-generation TDI combinations as given by Eqs. (4.5) and (4.12). This results in the following expression for ${\rho_2}^{\mathrm{GW,N}}$:

$$\begin{aligned} {\rho_2}^{\mathrm{GW,N}} = {} & \lambda_{\alpha_2} P_\alpha \alpha^{\mathrm{GW,N}} + \lambda_{\beta_2} P_\beta \beta^{\mathrm{GW,N}} + \lambda_{\gamma_2} P_\gamma \gamma^{\mathrm{GW,N}} \\ & + \lambda_{X_2} P_X X^{\mathrm{GW,N}}, \end{aligned} \tag{5.3}$$

where P_α (with P_β, P_γ obtained from it by permutations of the spacecraft indices) and P_X can be derived from Eqs. (4.12) and (4.5) relating the GW signal and the secondary noises of the second-generation combinations to their first-generation counterparts [21]. If we now select $(\lambda_{\alpha_2}, \lambda_{\beta_2}, \lambda_{\gamma_2}, \lambda_{X_2})$ to be equal to the following expressions,

$$\begin{aligned} \lambda_{\alpha_2} &= Q \lambda_\alpha P_\beta P_\gamma P_X, \\ \lambda_{\beta_2} &= Q \lambda_\beta P_\alpha P_\gamma P_X, \\ \lambda_{\gamma_2} &= Q \lambda_\gamma P_\alpha P_\beta P_X, \\ \lambda_{X_2} &= Q \lambda_X P_\alpha P_\beta P_\gamma \end{aligned} \tag{5.4}$$

(with Q being an arbitrary polynomial of the six delay operators) it is easy to derive the final expressions for ρ_2 and ${\rho_2}^{\mathrm{GW,N}}$ in terms of $\rho^{\mathrm{GW,N}}$ by substituting Eq. (5.4) in Eq. (5.3),

$$\begin{aligned} \rho_2 = {} & Q[\lambda_\alpha P_\beta P_\gamma P_X \alpha_2 + \lambda_\beta P_\alpha P_\gamma P_X \beta_2 + \lambda_\gamma P_\alpha P_\beta P_X \gamma_2 \\ & + \lambda_X P_\alpha P_\beta P_\gamma X_2], \end{aligned} \tag{5.5}$$

$${\rho_2}^{\mathrm{GW,N}} = Q P_\alpha P_\beta P_\gamma P_X \rho^{\mathrm{GW,N}}. \tag{5.6}$$

Equation (5.6) tells us that (i) the sensitivity of ρ_2 is identical to that of ρ because the Fourier components of the GW signals and the secondary noises in ρ_2 have the same transfer function to the GW signal and the secondary noises in ρ; (ii) the arbitrariness of the polynomial Q implies the mapping from the first-generation TDI space to the higher one is not one-to-one. In other words, lifting allows us to derive an infinite number of elements of the second-generation TDI space characterized by having the same GW sensitivity as their first-generation counterpart.

B. The Sagnac combination ζ

As an application of the above general result, we will now derive the expression for the second generation Sagnac combination ζ_2 that exactly cancels the laser noise up to linear velocity terms. In the case of almost equilateral arrays (with LISA being the most well-known example) among all TDI combinations the symmetric Sagnac ζ is characterized by a GW transfer function that suppresses the GW signal in the lower part of the accessible frequency band. By being still affected by the instrumental noise sources, ζ has been shown to provide future space-based GW interferometers with the capability of calibrating their in-flight noise performance in the presence of a strong astrophysical GW background [22].

Expressions for ζ that could exactly cancel the laser noise in the case of a rigidly rotating array were found in the literature [8,9]. They have also been shown to adequately suppress the laser noise below the secondary noise sources in the case of slowing varying armlengths. Here we show that it is possible to derive a family of ζ-like combinations that exactly cancel the laser noise up to linear velocity terms by taking specific linear combinations of (α_2, β_2, γ_2, X_2). Let us first write the following general linear combination of $(\alpha_2, \beta_2, \gamma_2, X_2)$,

$$\zeta_2 \equiv \lambda_X X_2 + \lambda_\alpha \alpha_2 + \lambda_\beta \beta_2 + \lambda_\gamma \gamma_2, \tag{5.7}$$

where the four polynomials of the delay operators, (λ_X, λ_α, λ_β, λ_γ) are at the moment unknown.

Since $(\alpha_2, \beta_2, \gamma_2, X_2)$ cancel exactly the laser noises, it is clear that any linear combination of them [such as that given in Eq. (5.7)] is also laser noise-free. Since (α_2, β_2, γ_2, X_2) now only contain the GW signal and the secondary noises, we can replace in Eq. (5.7) their expressions in terms of the first-generation TDI combinations as given by Eqs. (4.5) and (4.12). This results in the following expression for $\zeta_2^{\mathrm{GW,N}}$:

$$\begin{aligned}\zeta_2^{\mathrm{GW,N}} = {}& \lambda_X (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2) X^{\mathrm{GW,N}} \\ & + (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I) \\ & \times [\lambda_\alpha \alpha^{\mathrm{GW,N}} + \lambda_\beta \beta^{\mathrm{GW,N}} + \lambda_\gamma \gamma^{\mathrm{GW,N}}].\end{aligned} \tag{5.8}$$

Since $\zeta^{\mathrm{GW,N}} = \mathcal{D}_1 X^{\mathrm{GW,N}} - \mathcal{D}_2\mathcal{D}_3 \alpha^{\mathrm{GW,N}} + \mathcal{D}_2 \beta^{\mathrm{GW,N}} + D_3 \gamma^{\mathrm{GW,N}}$, it is then easy to identify the following expressions for the polynomials (λ_X, λ_α, λ_β, λ_γ) that guarantee ζ_2 to have the same sensitivity as ζ:

$$\begin{aligned}\lambda_X &= (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)\mathcal{D}_1, \\ \lambda_\alpha &= -(I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)\mathcal{D}_2\mathcal{D}_3, \\ \lambda_\beta &= (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)\mathcal{D}_2, \\ \lambda_\gamma &= (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)\mathcal{D}_3.\end{aligned} \tag{5.9}$$

Note the above four polynomials are not unique as they are defined up to an arbitrary polynomial multiplying them. If we now take the above expressions for (λ_X, λ_α, λ_β, λ_γ) and substitute them into Eq. (5.7) we obtain the final expressions for ζ_2 and $\zeta_2^{GW,N}$:

$$\begin{aligned}\zeta_2 = {}& (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)\mathcal{D}_1 X_2 \\ & + (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)[-\mathcal{D}_2\mathcal{D}_3\alpha_2 + \mathcal{D}_2\beta_2 + \mathcal{D}_3\gamma_2],\end{aligned} \tag{5.10}$$

$$\begin{aligned}\zeta_2^{GW,N} = {}& (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I) \\ & \times (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)\zeta^{GW,N}.\end{aligned} \tag{5.11}$$

As expected from the criterion adopted for identifying the four polynomials (λ_X, λ_α, λ_β, λ_γ), Eq. (5.11) explicitly shows that ζ_2 has the same sensitivity to GWs as ζ. This is because the Fourier components of the GW signals and the secondary noises in ζ_2 have the same transfer function to the GW signal and the secondary noises in ζ.

C. The monitor E_2 combinations

The monitor is a TDI combination that relies on only four Doppler measurements. As the name suggests, it corresponds to an array configuration in which one spacecraft can only receive laser light from the other two. To derive the second-generation TDI expression for such configuration, we first remind the reader that the first-generation TDI combination E is related to the basis elements (α, β, γ, X) through the relationship [3]

$$\begin{aligned}E &= \alpha - \mathcal{D}_1\zeta = \alpha - \mathcal{D}_1(\mathcal{D}_1 X - \mathcal{D}_2\mathcal{D}_3\alpha + \mathcal{D}_2\beta + \mathcal{D}_3\gamma) \\ &= -\mathcal{D}_1\mathcal{D}_1 X + (I + \mathcal{D}_1\mathcal{D}_2\mathcal{D}_3)\alpha - \mathcal{D}_1\mathcal{D}_2\beta - \mathcal{D}_1\mathcal{D}_3\gamma,\end{aligned} \tag{5.12}$$

where we have substituted the expression for ζ in terms of $(\alpha, \beta, \gamma, X)$ given in Eq. (2.3).

Let us now write again the following general linear combination of (α_2, β_2, γ_2, X_2)

$$E_2 \equiv \mu_X X_2 + \mu_\alpha \alpha_2 + \mu_\beta \beta_2 + \mu_\gamma \gamma_2, \tag{5.13}$$

where the four polynomials of the delay operators $(\mu_X, \mu_\alpha, \mu_\beta, \mu_\gamma)$ are unknown.

Since $(\alpha_2, \beta_2, \gamma_2, X_2)$ cancel exactly the laser noises, any linear combination of them [such as that given in Eq. (5.13)] is also laser noise-free. Since $(\alpha_2, \beta_2, \gamma_2, X_2)$ now only contain the GW signal and the secondary noises, we can replace in Eq. (5.13) their expressions in terms of the first-generation TDI combinations as given by Eqs. (4.5) and (4.12). This results in the following expression for $E_2^{\mathrm{GW,N}}$:

$$\begin{aligned}E_2^{\mathrm{GW,N}} = {}& \mu_X (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2) X^{\mathrm{GW,N}} \\ & + (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I) \\ & \times [\mu_\alpha \alpha^{\mathrm{GW,N}} + \mu_\beta \beta^{\mathrm{GW,N}} + \mu_\gamma \gamma^{\mathrm{GW,N}}].\end{aligned} \tag{5.14}$$

Since $E^{\mathrm{GW,N}} = -\mathcal{D}_1\mathcal{D}_1 X^{\mathrm{GW,N}} + (I + \mathcal{D}_1\mathcal{D}_2\mathcal{D}_3)\alpha^{\mathrm{GW,N}} - \mathcal{D}_1\mathcal{D}_2\beta^{\mathrm{GW,N}} - \mathcal{D}_1\mathcal{D}_3\gamma^{\mathrm{GW,N}}$, it is then easy to derive the following expressions for $(\mu_X, \mu_\alpha, \mu_\beta, \mu_\gamma)$ that guarantee E_2 to have the same sensitivity as E:

$$\begin{aligned}\mu_X &= -(\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)\mathcal{D}_1\mathcal{D}_1, \\ \mu_\alpha &= (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)(I + \mathcal{D}_1\mathcal{D}_2\mathcal{D}_3), \\ \mu_\beta &= -(I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)\mathcal{D}_1\mathcal{D}_2, \\ \mu_\gamma &= -(I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)\mathcal{D}_1\mathcal{D}_3.\end{aligned} \tag{5.15}$$

As in the case of ζ_2, the four polynomials identifying E_2 are also not unique as they are defined up to an arbitrary polynomial multiplying them. In other words, there exist an infinite number of monitor combinations in the second-generation TDI space.

If we now substitute the expressions above for the polynomials $(\mu_X, \mu_\alpha, \mu_\beta, \mu_\gamma)$ into Eq. (5.13), we obtain the following expressions for E_2 and $E_2^{GW,N}$:

$$E_2 = -(\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)\mathcal{D}_1\mathcal{D}_1 X_2 + (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)[(I + \mathcal{D}_1\mathcal{D}_2\mathcal{D}_3)\alpha_2 - \mathcal{D}_1\mathcal{D}_2\beta_2 - \mathcal{D}_1\mathcal{D}_3\gamma_2], \tag{5.16}$$

$$E_2^{GW,N} = (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I) \times (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)E^{GW,N}. \tag{5.17}$$

D. The beacon P_2 combinations

The beacon, like the monitor, is a TDI combination that relies on only four Doppler measurements. As the name suggests, it corresponds to an array configuration in which one spacecraft can only transmit laser light to the other two but is unable to receive from them. As in the case of the monitor combination, we first observe that the first-generation TDI combination P is related to the basis elements $(\alpha, \beta, \gamma, X)$ through the following relationship [3]:

$$\begin{aligned} P &= \zeta - \mathcal{D}_1\alpha \\ &= \mathcal{D}_1 X - \mathcal{D}_2\mathcal{D}_3\alpha + \mathcal{D}_2\beta + \mathcal{D}_3\gamma - \mathcal{D}_1\alpha \\ &= \mathcal{D}_1 X - (D_1 + \mathcal{D}_2\mathcal{D}_3)\alpha + \mathcal{D}_2\beta + \mathcal{D}_3\gamma, \end{aligned} \tag{5.18}$$

where again we have taken advantage of the expression for ζ in terms of $(\alpha, \beta, \gamma, X)$ given in Eq. (2.3).

As was done in the previous subsection, we first take a linear combination of $(\alpha_2, \beta_2, \gamma_2, X_2)$ with four unknown polynomials $(\nu_X, \nu_\alpha, \nu_\beta, \nu_\gamma)$ of the delay operators

$$P_2 \equiv \nu_X X_2 + \nu_\alpha \alpha_2 + \nu_\beta \beta_2 + \nu_\gamma \gamma_2. \tag{5.19}$$

Since $(\alpha_2, \beta_2, \gamma_2, X_2)$ cancel exactly the laser noises, any linear combination of them [such as that given by Eq. (5.19)] is also laser noise-free. This implies that we can replace in Eq. (5.19) their expressions in terms of the first-generation TDI combinations as given by Eqs. (4.5) and (4.12). This results in the following expression for $P_2^{\mathrm{GW,N}}$:

$$P_2^{\mathrm{GW,N}} = \nu_X(I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)X^{\mathrm{GW,N}} + (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I) \times [\nu_\alpha \alpha^{\mathrm{GW,N}} + \nu_\beta \beta^{\mathrm{GW,N}} + \nu_\gamma \gamma^{\mathrm{GW,N}}]. \tag{5.20}$$

Since $P^{\mathrm{GW,N}} = \mathcal{D}_1 X^{\mathrm{GW,N}} - (\mathcal{D}_1 + \mathcal{D}_2\mathcal{D}_3)\alpha^{\mathrm{GW,N}} + \mathcal{D}_2\beta^{\mathrm{GW,N}} + \mathcal{D}_3\gamma^{\mathrm{GW,N}}$, it is then easy to recognize the following expressions for $(\nu_X, \nu_\alpha, \nu_\beta, \nu_\gamma)$ guarantee P_2 to have the same sensitivity as P:

$$\begin{aligned} \nu_X &= (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)\mathcal{D}_1, \\ \nu_\alpha &= -(I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)(\mathcal{D}_1 + \mathcal{D}_2\mathcal{D}_3), \\ \nu_\beta &= (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)\mathcal{D}_2, \\ \nu_\gamma &= (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)\mathcal{D}_3. \end{aligned} \tag{5.21}$$

The above four polynomials are again defined up to an arbitrary polynomial multiplying them. By substituting them into Eq. (5.19) we finally get

$$P_2 = (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)\mathcal{D}_1 X_2 + (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)[-(D_1 + \mathcal{D}_2\mathcal{D}_3)\alpha_2 + \mathcal{D}_2\beta_2 + \mathcal{D}_3\gamma_2], \tag{5.22}$$

and

$$P_2^{GW,N} = (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I) \times (I - \mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2)P^{GW,N}. \tag{5.23}$$

E. The relay U_2 combinations

The relay is a TDI combination corresponding to an array configuration in which one spacecraft can only receive along one arm and transmit along the other. As in the case of the previous two four-link combinations, we first observe that the first-generation TDI combination U is related to the basis elements $(\alpha, \beta, \gamma, X)$ through the following relationship [3]:

$$U = \mathcal{D}_1\gamma - \beta. \tag{5.24}$$

Given the above form of U, the most general expression for U_2 will be determined by the following linear combination of β_2 and γ_2:

$$U_2 \equiv \delta_\beta \beta_2 + \delta_\gamma \gamma_2, \tag{5.25}$$

where δ_β, δ_γ are unknown polynomials of the delay operators. Since (β_2, γ_2) cancel exactly the laser noises, any linear combination of them [such as that given by Eq. (5.25)] is also laser noise-free. This implies that we can replace in Eq. (5.25) their expressions in terms of the first-generation TDI combinations as given by Eq. (4.12). This results in the following expression for $U_2^{\mathrm{GW,N}}$:

$$U_2^{\mathrm{GW,N}} = (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I) \times [\delta_\beta \beta^{\mathrm{GW,N}} + \delta_\gamma \gamma^{\mathrm{GW,N}}]. \tag{5.26}$$

Since $U^{\mathrm{GW,N}} = -\beta^{\mathrm{GW,N}} + \mathcal{D}_1\gamma^{\mathrm{GW,N}}$, it is easy to identify the following expressions for $(\delta_\beta, \delta_\gamma)$ guarantee U_2 to have the same sensitivity of U:

$$\delta_\beta = -1, \quad \delta_\gamma = \mathcal{D}_1. \tag{5.27}$$

The above two polynomials are defined up to an arbitrary polynomial multiplying them. The resulting expressions for U_2 and $U_2^{GW,N}$ are therefore equal to

$$U_2 = -\beta_2 + D_1\gamma_2 \tag{5.28}$$

and

$$U_2^{GW,N} = (\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)(\mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'} - I)U^{GW,N}. \tag{5.29}$$

VI. CONCLUSIONS

We revisited the second-generation TDI space, i.e. the set of TDI combinations canceling the laser noise up to terms linear in the time derivatives of the interspacecraft light travel times. We identified analytic expressions for the Sagnac $(\alpha_2, \beta_2, \gamma_2)$ and unequal-arm Michelson combination X_2 that exactly cancel the laser noises up to linear terms in the interspacecraft velocities. Our derivation relies on an iterative procedure we named "lifting." This technique entails making two synthesized laser beams go around the array along clockwise and counterclockwise paths a number of times before interfering back at the transmitting spacecraft. We found that, to cancel the laser phase fluctuations (up to linear velocity terms) in the Sagnac combination α, the two synthesized beams need to make at least three loops around the array before interfering back at the transmitting spacecraft. By relying on the expressions of the lifted Sagnac $(\alpha_2, \beta_2, \gamma_2)$ and unequal-arm Michelson combinations, X_2, we were able to show that any element of the first-generation TDI space can be lifted up. In particular we were able to identify an infinite number of expressions for ζ-like, monitor, beacon, and relay combinations. This was done by taking linear combinations of $(\alpha_2, \beta_2, \gamma_2, X_2)$ with polynomials of the delay operators that result in TDI combinations whose sensitivities equal those of their first-generation counterparts. In this regard we can say of having identified a mapping between the first- and the second-generation TDI spaces by which any element of the first-generation TDI space is lifted up.

We believe the iterative procedure so effectively employed in this article may be extended to cancel the laser frequency noise at higher orders. We will follow up on these ideas in our forthcoming investigation.

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