Varied avatars of time-delay interferometry

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Time-delay interferometry (TDI) is a data processing technique that cancels the large laser phase fluctuations affecting the one-way Doppler measurements made by unequal-arm space-based gravitational wave interferometers. By taking finite linear combinations of properly time-shifted Doppler measurements, laser phase fluctuations can be removed at any time t and gravitational wave signals can be studied at the requisite level of sensitivity. In the past, other approaches to this problem have been proposed. Recently, matrix-based approaches have been put forward; two such approaches are by Vallisneri *et al.* and Tinto *et al.* In this paper, we establish a close relationship between these approaches. In fact, we show that the matrices involved in defining the operators in the two approaches exhibit an isomorphism, and therefore, in both approaches one is dealing with matrix representations of the time-delay operators.

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I. INTRODUCTION

In ground-based detectors of gravitational waves (GWs) the arms are chosen to be of equal length. This is because the laser phase fluctuations experience identical delays in the arm of the interferometer and cancel at the photodetector, where the two returning beams are made to interfere. In space-based detectors, on the other hand, the arm lengths are unequal and time dependent, as each spacecraft follows a trajectory determined by celestial mechanics. As a result, it becomes impossible to maintain the distances between spacecraft equal and constant. Time-delay interferometry (TDI) is required to cancel the laser phase noise, which is many orders of magnitude above the other residual noise sources (such as shot noise, test mass acceleration noise, etc.) affecting the heterodyne one-way measurements. TDI entails properly delaying and linearly combining the different data streams so that the laser phase fluctuations are suppressed below the residual noises and GW signals may be observed.

In the past, other approaches have been proposed to compensate for the inequality of the arms and achieve suppression of the laser noise below the residual noise levels. The first, which was formulated in the Fourier domain [1], represented the delayed one-way measurements in terms of their Fourier transforms multiplied by corresponding phasors. This approach was incorrect for two fundamental reasons. First, it neglected the time evolution of the delays, which we now know needs to be accounted for to sufficiently suppress the laser noise. Second (and more important), it made the erroneous assumption of taking infinitely long Fourier transforms of the delayed one-way measurements. A finite-time Fourier transform of a delayed measurement is not equal to the product of its Fourier transform with the delay phasor. Rather, it is equal to the Fourier domain convolution of the Fourier transform of the data with the Fourier transform of the window of integration. This implied the existence of residual laser noise terms in the Fourier-domain laser-noise-canceling algorithm that could be neglected only by taking six months or longer Fourier transforms of the measurements [2].

A much neater approach by Tinto and co-workers, where the delays were represented by derivativelike symbols, namely, commas [2–5], was applied in the context of the Laser Interferometer Space Antenna

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(*LISA*) mission [6,7]. This notation (and its understanding) facilitated the algebra of the TDI observables and led, in principle, to a plethora of such observables that could be obtained conveniently by just linearly combining the four Sagnac observables α, β, γ, and ζ. This work showed that the space of TDI was a linear object and its elements could be obtained by linearly combining four simple basic TDI observables. This is, in fact, the most important result of this approach.

An exceptionally deep insight into time-delay interferometry was obtained, when Dhurandhar *et al.* [8] found the exact underlying mathematical structure of the TDI space. In this approach, the time-delay operation was promoted to operators acting on data streams or operators acting on functions of time. The operators played the role of indeterminates in a polynomial ring and the TDI space was none other than the first module of syzygies [9]. It was shown therein that the TDI space is a module over the polynomial ring of time-delay operators and hence pinned down the linear structure. This work laid emphasis on the operators rather than on the functions (data streams containing laser noise) since these were subsidiary—the function space is the carrier space. This is a similar situation as one has in matrix representations of groups; the matrices, which are linear maps on the carrier vector space, represent the group elements. This is interesting from the historical point of view because the first module of syzygies was defined by Hilbert in 1890 [10] in a different context. It is, in fact, a kernel of a homomorphism. This is exactly what one desires—its physical significance here is that elements in a kernel map to zero. In the current context, this is the zero of the laser noise: we are looking for those data combinations that map the laser phase noise to zero. This approach rigorously proved that all TDI observables can be obtained as a linear combination of the four generators α, β, γ, and ζ.[1] The first work [8] considered constant arm lengths. Later, more general and realistic models of *LISA* were considered [5,11–14], which increased the complexity. These approaches with generalizations have been reviewed comprehensively in [15].

Romano and Woan first came up with the idea of using matrices for TDI by employing the method of principal component analysis [16]. This idea was further investigated by Leighton in a Ph.D. thesis [17]. In fairly recent years, another matrix-based approach was adopted by Vallisneri *et al.* [18]. This method was formulated in the frequency domain by Baghi *et al.* [19,20] and also in a model-independent way. In this paper, we focus on the approach adopted by Vallisneri *et al.* in which the data are discretized and a design matrix representing the delays is constructed. However, since data points may be required in between the sample points for TDI to be effective, an interpolating scheme must be employed for fractional delays. Here also a null space is sought, whose elements are then the TDI observables. Another matrix approach was put forward by Tinto *et al.* [21]. In this work, it was shown that the matrix approach is a ring representation [22] of the operator approach—there is a homomorphism between the ring of operators into the ring of matrices. However, the matrix approach seems to have an advantage because matrices are easy to manipulate (although this has not been conclusively established). This is in the same spirit, as one uses group representations rather than abstract group elements to perform calculations. In this paper, we show that there is, in fact, an isomorphism (which is more than homomorphism—the map is also one to one and onto) between the design matrices defined in [18] and the matrix operators defined in our approach [21].

This paper is organized as follows. In Sec. II, we briefly describe the formulation in [18], which employs design matrices to pose the TDI problem. We extract half the rows corresponding to one arm because they have the basic structure that we wish to investigate. In order to make the paper as self-contained as possible, we recall results from [21] in which the homomorphism between the delay operators and the D matrices is presented. We also display a few D matrices for integer delays for concreteness. In Sec. III, we then prove the isomorphism between the matrices defined in [18,21], while in Sec. IV we show how to generalize the one-arm results to the two-arm configuration. In Sec. V, we finally present our concluding remarks and emphasize that the isomorphism existing between our matrix representation of the TDI delay operators and the matrices introduced in [18] should help us in relating the laser noise-free combinations identified by the two methods.

II. ALGEBRA OF DESIGN MATRICES: THE CASE OF THE SINGLE ARM

A. The design matrix formulation

We first briefly describe the scheme proposed in [18]. Here the sampled two two-way Doppler data are packaged in a single array in an alternating fashion starting from time $t = t_0$ when the laser is switched on. Assuming a stationary array configuration in which the round-trip delays denoted by l_1, l_2 are equal to 2 and 3 times the sampling time Δt (as exemplified in [18]), the measurements array is linearly related to the array associated with the samples of the laser noise c through a rectangular $2N \times N$ matrix M (N being the number of samples considered) in the following way:

[1] A module in general does not have a basis, but has generators which span the module though they may not be linearly independent—one may not be able to reduce the number of generators in general, because multiplicative inverses need not exist in a ring.

$$\begin{pmatrix} y_1(t_0) \\ y_2(t_0) \\ y_1(t_1) \\ y_2(t_1) \\ y_1(t_2) \\ y_2(t_2) \\ y_1(t_3) \\ y_2(t_3) \\ y_1(t_4) \\ y_2(t_4) \\ \vdots \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & -1 & 0 & \cdots \\ 1 & 0 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & -1 & \cdots \\ 0 & 1 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} c(t_0) \\ c(t_1) \\ c(t_2) \\ c(t_3) \\ c(t_4) \\ \vdots \end{pmatrix}. \tag{2.1}$$

The equation may be written more compactly as

$$\mathbf{y} = M\mathbf{c}, \tag{2.2}$$

where the measurement vector $\mathbf{y}$ is related to the laser noise vector $\mathbf{c}$ by Eq. (2.1) (we have ignored in this equation the noise other than the laser noise).

We will consider just one of the arms, say arm 1, in our discussion. We do not consider the interleaving of y_1 and y_2. In Sec. IV, we will indicate how to generalize the analysis to the case for two arms. We just consider the measurements y_1 whose components (or samples) appear in odd numbered rows of the column vector $\mathbf{y}$. Further, we will only consider that part of the matrix that describes the delay and therefore disregard the subtraction of the unit matrix; that is, if M is the design matrix for one arm, then we consider the matrix $V = M + I$, where I is the identity matrix. We denote the matrix by V since the design matrix has been introduced in the paper by Vallisneri *et al.* [18].

In the next subsection, we recall some results that were obtained in [21] on the homomorphism between the delay operators $\mathcal{D}$ and the corresponding D matrices.

B. The D matrices

In the continuum limit, the delayed data are given by $\mathcal{D}y(t) = y(t - l)$. As shown in [21], these operators form a ring and they can be mapped to matrices that preserve the ring operations—it is a ring homomorphism. We briefly discuss this below for integer delays, because it is easy to understand this intuitively. Consider a data segment of finite duration $[0, T]$. We will assume that the data are sampled uniformly with sampling time interval Δt. Now there are a finite number of samples N labeled by the times $t_k = k\Delta t, k = 0, 1, 2, \ldots, N - 1$ and also we have $N\Delta t = T$. Since we are considering only a single arm here, the measurements y (we drop the subscript "1" on y to avoid clutter) and the laser noise c can be represented by N-dimensional vectors in $\mathbf{y}, \mathbf{c} \in \mathcal{R}^N$. The operators $\mathcal{D}$ now take the form of linear transformations from $\mathcal{R}^N \to \mathcal{R}^N$ and hence in our formulation can be represented by $N \times N$ matrices, which now for this case we will represent by just D. We have essentially discretized the previous situation of the continuum. In the matrix representation, we have represented the abstract TDI operators $\mathcal{D}$ by the matrices D. The operations that were valid in the abstract case map faithfully to their discretized versions. The sum and product of the $\mathcal{D}$ operators map to the sum and product of the D matrices—the ring operations are preserved, and thus it is a ring homomorphism. This is, in fact, known as a matrix representation of a ring in the literature [22].

It is first of all easily verified that for integer delays the product of two operators, say, $\mathcal{D}_1$ and $\mathcal{D}_2$ corresponds to the product of the matrices D_1 and D_2. For simplicity, we will take the sampling interval to be unity. The delay matrices for these delays $\Delta t = 1, 2, 3$ have been explicitly displayed in [21]. For the sake of completeness and convenience, we also reproduce them here,

$$D_1 = \begin{pmatrix} 0&0&0&0&0&0&0&\cdots \\ 1&0&0&0&0&0&0&\cdots \\ 0&1&0&0&0&0&0&\cdots \\ 0&0&1&0&0&0&0&\cdots \\ 0&0&0&1&0&0&0&\cdots \\ 0&0&0&0&1&0&0&\cdots \\ 0&0&0&0&0&1&0&\cdots \\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots \end{pmatrix} \quad D_2 = \begin{pmatrix} 0&0&0&0&0&0&0&\cdots \\ 0&0&0&0&0&0&0&\cdots \\ 1&0&0&0&0&0&0&\cdots \\ 0&1&0&0&0&0&0&\cdots \\ 0&0&1&0&0&0&0&\cdots \\ 0&0&0&1&0&0&0&\cdots \\ 0&0&0&0&1&0&0&\cdots \\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots \end{pmatrix} \quad D_3 = \begin{pmatrix} 0&0&0&0&0&0&0&\cdots \\ 0&0&0&0&0&0&0&\cdots \\ 0&0&0&0&0&0&0&\cdots \\ 1&0&0&0&0&0&0&\cdots \\ 0&1&0&0&0&0&0&\cdots \\ 0&0&1&0&0&0&0&\cdots \\ 0&0&0&1&0&0&0&\cdots \\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots \end{pmatrix}. \tag{2.3}$$

It is easily verified that $D_3 = D_1 D_2$.

The V matrices for integer delays can be taken to be identical to the D matrices and so the homomorphism for the V matrices follows as shown in [21]. This disposes of the integer delays.

For constant fractional delays, we only need to exhibit a bijective map between the matrices D defined in [21] and the matrices V. First, we define the D matrices for fractional delays. Following [18] we use Lagrange polynomials for interpolation. The general expression for the Lagrange polynomials on m nodes at $t_j, j = 0, 1, \ldots, m-1$ is

$$l_j(t) = \frac{(t-t_0)(t-t_1)\ldots(t-t_{j-1})(t-t_{j+1})\ldots(t-t_{m-1})}{(t_k-t_0)\ldots(t_k-t_{j-1})(t_j-t_{j+1})\ldots(t_j-t_{m-1})}, \quad j = 0, 1, \ldots, m-1. \tag{2.4}$$

The nodes are the time samples where the data are measured. We use these discretely measured data to interpolate at points other than the nodes by using Lagrange interpolation. Consider the interval $I_0 = \{0, 1, 2, \ldots, m-1\}$, which accommodates all delays. Here $t_j = j, j = 0, 1, \ldots, m-1$. Consider a fractional delay α. As argued in [21], the operator $\mathcal{D}(\alpha)$ is represented by the $m \times m$ matrix,

$$D_{jk}(\alpha) = l_k(\alpha + j). \tag{2.5}$$

If we consider two such delays α and β, then $\mathcal{D}(\alpha)\mathcal{D}(\beta) = \mathcal{D}(\alpha+\beta)$ and as proved in [21] this operation is reflected faithfully by the corresponding matrices, and hence the homomorphism is established for fractional delays.

A more general case that will turn out to be useful here is when the delays α and β and $\alpha+\beta$ do not lie in a single interval. Then we have, in general, three intervals, $I_r = \{r, r+1, \ldots, r+m-1\}, I_s = \{s, s+1, \ldots, s+m-1\}$, and $I_{r+s} = \{r+s, r+s+1, \ldots, r+s+m-1\}$ around α, β and $\alpha+\beta$, respectively. Let $l_j(t), j = 0, 1, \ldots, m-1$ be the Lagrange polynomials for the reference interval $I_0 = \{0, 1, 2, \ldots, m-1\}$. Then the Lagrange polynomials for the interval I_r are just the translated versions of $l_j(t)$, namely, $l_j(t-r)$ and similarly $l_j(t-s)$ for I_s. In this case, the translated matrix representation is $D^{(r)}_{jk}(\alpha) = l_k(\alpha - r + j)$ for delay α and $D^{(s)}_{jk}(\beta) = l_k(\beta - s + j)$ for β. By a judicious choice of r and s, we can make $\alpha + \beta \le r + s + m - 1$ so that the relevant interval is I_{r+s}. The homomorphism is given by

$$\begin{aligned}\sum_k D^{(r)}_{jk}(\alpha) D^{(s)}_{kn}(\beta) &= \sum_k l_k(\alpha - r + j) l_n(\beta - s + k) \\ &\equiv l_n(\alpha + \beta - (r+s) + j) \\ &= D^{(r+s)}_{jn}(\alpha+\beta).\end{aligned} \tag{2.6}$$

This equation was derived in [21], namely, Eq. (4.15) therein, where the addition theorem for Lagrange polynomials was used.

C. The V matrices

We now turn to the structure of the matrix V for fractional delays. The Lagrange polynomial interpolation scheme is chosen as in [18] with the degree m of the polynomials set equal to 6. The data y are labeled at integer nodes at $t_j = j, j = 0, 1, 2, \ldots$ and are denoted accordingly by $y_j = y(t_j)$. Now consider a fractional delay α. The noninteger delay α is broken up into its integer part $[\alpha]$ and the residual part $\delta\alpha$ as $\alpha = [\alpha] + \delta\alpha$. Now the interval containing m nodes has to be chosen so that it covers the delayed time instant and such that it lies somewhere near the center of the interval. This depends at what time instant we are evaluating the delayed data. If the time instant is j, then we go back $[\alpha] + m/2 = [\alpha] + 3$ nodes, and it is at this node the filter mask starts. For example, if $\alpha = 2.2$, then $[\alpha] = 2$ and one must start the interpolating interval from $j - 5$. To fix ideas, we will use this value of α to explain the structure of V and later mention how this generalizes to any value of α. If the data are measured from $t = 0$, then a full mask is possible only when $j \ge 5$. The first such instant occurs at $j = 5$ and the interval is $\{0, 1, 2, 3, 4, 5\}$ with data values $\{y_0, y_1, \ldots, y_5\}$ at the corresponding nodes. The interpolated data are evaluated at $t = 5 - \alpha = 2.8 \equiv \alpha'$, say, or $y(2.8)$. In general, $\alpha' = m/2 - \delta\alpha$, ensuring that the interpolation point is near to the center of the filter mask. Note that $l_j(t)$ are polynomials of degree $m - 1$. Here we have $m = 6$ and so the polynomials are of degree 5. Then the interpolated value of y at $t = \alpha'$ is given by

$$y(\alpha') = \sum_{j=0}^{5} l_j(\alpha') y_j. \tag{2.7}$$

In the V matrix, written as V_{jk}, the first row with six nonzero entries occurs first at the $j = 5$ row (this is the sixth row since the index j runs from $0, 1, 2, \ldots$ Then we have $V_{5k} = l_k(\alpha')$ for $0 \le k \le 5$ and $V_{5k} = 0$ for $k > 5$. In the next row $j = 6$, the filter mask covers the interval $\{1, 2, \ldots, 6\}$, and the corresponding data points are $\{y_1, y_2, \ldots, y_6\}$. In the matrix V, the Lagrange polynomials are shifted by one column to the right, where we have $V_{60} = 0, V_{6k} = l_{k-1}(\alpha'), k = 1, 2, \ldots, 6$, and $V_{6k} = 0$ for $k > 6$. Here, for simplicity, we have chosen the time-delay to be constant (the time-dependent case does not make much difference to the homomorphism argument). As one proceeds down the rows, the Lagrange polynomials get shifted to the right and so diagonally downward. The matrix V looks as follows:

$$V(\alpha') = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ l_0(\alpha') & l_1(\alpha') & l_2(\alpha') & l_3(\alpha') & l_4(\alpha') & l_5(\alpha') & 0 & 0 & 0 & \cdots \\ 0 & l_0(\alpha') & l_1(\alpha') & l_2(\alpha') & l_3(\alpha') & l_4(\alpha') & l_5(\alpha') & 0 & 0 & \cdots \\ 0 & 0 & l_0(\alpha') & l_1(\alpha') & l_2(\alpha') & l_3(\alpha') & l_4(\alpha') & l_5(\alpha') & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.8}$$

For $\alpha' = 2.8$, the l_k take the numerical values $0.006336, -0.04928, 0.22176, 0.88704, -0.07392$, and 0.008064 as k ranges from 0 to 5 in steps of unity.

III. THE ISOMORPHISM BETWEEN V AND D MATRICES

For establishing the homomorphism, this arrangement presents difficulties because the target subspace changes (advances) with each successive row. We will continue taking $\alpha = 2.2$. For $j = 5$, the target subspace is the interval $W_0 = \{0, 1, 2, 3, 4, 5\}$, while for $j = 6$, the target subspace is $W_1 = \{1, 2, 3, 4, 5, 6\}$ and so on. In order to establish homomorphism, one requires a fixed target subspace. In the language of group representation theory, the target space is the carrier space. Given a group $\mathcal{G}$, a group element $g \in \mathcal{G}$ is mapped to linear transformation T_g, where $T_g: W \to W$, where W is a finite-dimensional vector space. The linear transformation T_g is then represented by a matrix and a product of two group elements is mapped to a product of the two corresponding matrices. Thus, instead of working with abstract group elements, one may work with the corresponding matrices, which facilitates computations. The vector space W is called a carrier space. We therefore fix a target subspace. There are several choices for this; we make the following one. We fix the target subspace to be W_0 and so refer all the Lagrange polynomials to W_0; in effect, we translate the Lagrange polynomials to W_0. This means we leave the row $j = 5$ unaltered because the target space is W_0. For the next row, $j = 6$, we need to shift by one column to the left in order to obtain the same target space W_0. This is achieved by translating the Lagrange polynomials—that is, by adding 1 to the argument α'. This makes sense because, for the value of $\alpha' = 2.8$, we evaluate the Lagrange polynomial at $\alpha' + 1 = 3.8 = 6 - 2.2 = 6 - \alpha$. Thus, the translated Lagrange polynomials are $l_k(\alpha' + 1)$. Similarly, for the next row $j = 7$, one needs to shift the Lagrange polynomials by two columns to the left, and therefore we must add 2 to the argument α' resulting in $l_k(\alpha' + 2)$. Thus, the entries in rows $j = 5$–10 are shifted to the left by the appropriate number of columns, with the arguments of the Lagrange polynomials increased by the number equal to the number of shifted columns. We only take the rows $5 \leq j \leq 10$ of the V matrix because our purpose is to obtain a 6×6 square block in the translated matrix, which we call V^{trans} and make it block diagonal with identical blocks. The $j = 10$ row has nonzero entries only up to the column $k = 10$, and therefore the block of V we consider is a 6×11 matrix, where $5 \leq j \leq 10$ and $0 \leq k \leq 10$. This block we call B_1 or the first block of V. By carrying out this procedure, we obtain a 6×6 translated block B_1^{trans} of the matrix V^{trans} given below,

$$V^{\mathrm{trans}}(\alpha') = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ l_0(\alpha') & l_1(\alpha') & l_2(\alpha') & l_3(\alpha') & l_4(\alpha') & l_5(\alpha') & 0 & 0 & 0 & \cdots \\ l_0(\alpha'+1) & l_1(\alpha'+1) & l_2(\alpha'+1) & l_3(\alpha'+1) & l_4(\alpha'+1) & l_5(\alpha'+1) & 0 & 0 & 0 & \cdots \\ l_0(\alpha'+2) & l_1(\alpha'+2) & l_2(\alpha'+2) & l_3(\alpha'+2) & l_4(\alpha'+2) & l_5(\alpha'+2) & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & \cdots \\ l_0(\alpha'+5) & l_1(\alpha'+5) & l_2(\alpha'+5) & l_3(\alpha'+5) & l_4(\alpha'+5) & l_5(\alpha'+5) & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.1}$$

First, for all rows of B_1^{trans}, the target space is fixed and it is W_0, and so we have achieved our goal. Next, we immediately recognize that the above 6×6 block matrix B_1^{trans} of the Lagrange polynomials is identical with the D matrix of Eq. (2.5) for $m = 6$ and $\alpha = \alpha'$. As shown in [21], the D matrices form a representation of the fractional delay operators, and hence it follows that the block matrices B_1^{trans} in V^{trans} also constitute a representation of the $\mathcal{D}$ operators.

The above discussion was relevant to the first 6×6 block $B_1^{\rm trans}$ of $V^{\rm trans}$ obtained from B_1. The next 6×6 block of $V^{\rm trans}$, namely, $B_2^{\rm trans}$ is obtained by carrying out the same procedure as above on the next block B_2 of V. The B_2 block of V consists of rows $11 \le j \le 16$ and columns $11 \le k \le 21$. The corresponding $B_2^{\rm trans}$ block consists of rows $11 \le j \le 16$ and columns $6 \le k \le 11$. Thus, the target subspace is fixed to be W_6 for $B_2^{\rm trans}$. Thus, $B_2^{\rm trans}$ is identical to $B_1^{\rm trans}$. This procedure is continued down the rows to obtain 6×6 blocks of identical square matrices. This makes $V^{\rm trans}$ block diagonal, each block a 6×6 matrix. The homomorphism can be extended to whole of $V^{\rm trans}$ in an obvious way.

Since we had chosen $\alpha = 2.2$ and $m = 6$, the blocks began at the row $j = [\alpha] + 3 = 5$. In general, the blocks begin at the row $j = [\alpha] + m/2$ for a filter mask of m nodes. Thus, an analogous procedure as carried out above may be followed to obtain $V^{\rm trans}$ from V. $V^{\rm trans}$ is still block diagonal, each block $m \times m$, where the blocks begin at the row $j = [\alpha] + m/2$.

We can also perform the operation of shifting the rows. This is just translating the polynomials by the required time stamps. If we shift the rows "upward" by r time samples, then we must subtract r from the argument of the Lagrange polynomials. For example, in the above example, if we shift by two time samples upward, the arguments in any column of the 6×6 block will range from $\alpha' - 2$ to $\alpha' + 3$. Thus, in our case of $\alpha' = 2.8$, the arguments will range from 0.8 to 5.8 close to the interpolation nodes of W_0. This is relevant when numerical accuracy is a consideration. The opposite happens if we shift down and to the right—shifting diagonally downward and to the right (or upward and to the left) keeps the entries in each row the same—the arguments of the Lagrange polynomials do not change. We have assumed here constant arm lengths for simplicity. We will remark later on time-dependent arm lengths; they do not cause any difficulty, in principle, to the homomorphism argument.

We can describe this representation directly, in general, for a filter mask on m nodes. Let α and β be two time delays. Consider first the first block of the corresponding $V^{\rm trans}$ matrices. Define $r = [\alpha] + m/2$ and $s = [\beta] + m/2$, then $V^{\rm trans}_{r+j,k}(\alpha') = l_k(\alpha' + j)$ and $V^{\rm trans}_{s+k,n}(\beta') = l_n(\beta' + k)$, where $j, k, n = 0, 1, \ldots m - 1$, $\alpha' = r - \alpha$, and $\beta' = s - \beta$. Then we obtain

$$\sum_{k=0}^{m-1} V^{\rm trans}_{r+j,k}(\alpha') V^{\rm trans}_{s+k,n}(\beta') = V^{\rm trans}_{r+s+j,n}(\alpha' + \beta'). \tag{3.2}$$

The above equation follows from the addition theorem for Lagrange polynomials [21]. Equation (3.2) directly exhibits the homomorphism restricted to the first blocks. This is, in fact, the same as equation Eq. (2.6). Since $m/2$ is added both to r and s, we may shift the rows by $m/2$ upward by writing the rhs of Eq. (3.2) as $V^{\rm trans}_{r+s+j-m/2,n}(\alpha' + \beta' - m/2)$ to get it in the required form as given in [18]. This rule is obtained if $\delta\alpha + \delta\beta < 1$. If this is not the case, that is, if $\delta\alpha + \delta\beta \ge 1$, then the required upward shift is $m/2 - 1$. This matrix is $V^{\rm trans}(\alpha' + \beta')$. We write as follows:

$$V^{\rm trans}(\alpha' + \beta') = \begin{cases} V^{\rm trans}_{r+s+j-m/2,n}(\alpha' + \beta' - m/2), & \delta\alpha + \delta\beta < 1, \\ V^{\rm trans}_{r+s+j-m/2+1,n}(\alpha' + \beta' - m/2 + 1), & \delta\alpha + \delta\beta \ge 1. \end{cases} \tag{3.3}$$

The second block $B_2^{\rm trans}$ is obtained by moving the first block diagonally downward by m rows and columns. This is formally achieved by replacing r, s, k, and n by $r + m, s + m, k + m$, and $n + m$, respectively. So we have for the second block the component matrices $V^{\rm trans}_{r+m+j,k+m}(\alpha')$ and $V^{\rm trans}_{s+m+k,n+m}(\beta')$ whose composition ("product") is $V^{\rm trans}_{r+s+j+m/2,n+m}(\alpha' + \beta' + m/2 + \epsilon)$, where $\epsilon = 0$ if $\delta\alpha + \delta\beta < 1$ and $\epsilon = 1$ otherwise. We note that the product matrix has its rows and columns shifted by m as it should be and the argument is also increased by m because this block acts on W_6. $B_2^{\rm trans}$ is identical to $B_1^{\rm trans}$ as an $m \times m$ matrix but only moved diagonally downward by m rows and m columns. We can continue in this way to obtain the remaining blocks and therefore extend the homomorphism to entire $V^{\rm trans}$. We therefore formally write

$$V^{\rm trans}(\alpha') \star V^{\rm trans}(\beta') = V^{\rm trans}(\alpha' + \beta'). \tag{3.4}$$

We have denoted the composition operation by $\star$.

The time-dependent case follows exactly the discussion in [21]. If the delay β is applied after α, then β becomes a function of α and the composite delay is given by $\alpha + \beta(\alpha)$. If the order of the delays is reversed, then the composite delay is $\beta + \alpha(\beta) \neq \alpha + \beta(\alpha)$. Thus, the delay operators and their representative matrices do not commute, in general. This is the basic difference between the time-dependent and time-independent cases. The rest of the discussion parallels the discussion for the time-independent case.

The only point remaining is to formally establish the correspondence between V and $V^{\rm trans}$. Let this correspondence be denoted by the mapping ψ. We show below that ψ

is in fact an isomorphism (not merely a homomorphism). Thus, we write

$$\psi[V(\alpha')] = V^{\mathrm{trans}}(\alpha'). \tag{3.5}$$

If we show that ψ^{-1} exists, then the composition law for the V matrices, described by the operation $\star'$, can be obtained through ψ^{-1} and $\star$ because

$$\begin{aligned} V(\alpha')\star' V(\beta') &= \psi^{-1}\{\psi[(V(\alpha')]\star\psi[V(\beta')]\} \\ &= \psi^{-1}[V^{\mathrm{trans}}(\alpha')\star V^{\mathrm{trans}}(\beta')] \\ &= \psi^{-1}[V^{\mathrm{trans}}(\alpha'+\beta')] = V(\alpha'+\beta'). \end{aligned} \tag{3.6}$$

We can describe the operation $\star'$ as follows. Given delays α and β, we have the matrices $V(\alpha')$ and $V(\beta')$. Using ψ, we translate them to $V^{\mathrm{trans}}(\alpha') = \psi(V(\alpha'))$ and $V^{\mathrm{trans}}(\beta') = \psi(V(\beta'))$. We then carry out the composition of these matrices using $\star$ and hence obtain $V^{\mathrm{trans}}(\alpha'+\beta')$. Then we use ψ^{-1} to pull back $V^{\mathrm{trans}}(\alpha'+\beta')$ to $V(\alpha'+\beta')$.

Here some remarks are in order. It is first of all clear that ψ is linear and hence a homomorphism. We will establish below that ψ is bijective and hence an "isomorphism." We will also explicitly establish this fact by exhibiting formulas. In the literature [22], the map ψ or, more appropriately, its extension $\tilde{\psi}$ is called an "intertwinor" and the representation is an intertwining representation. Thus, the matrices V form an intertwining representation. In the Appendix, we indicate how the map ψ is extended to the intertwining map $\tilde{\psi}$.

We take $m = 6$ in order to elucidate our arguments. The arguments may be easily generalized to a general value of m. Consider the first block B_1 of V. It is easily shown that the six row vectors of the block matrix, namely, $R_k, k = 0, 1, \ldots, 5$ are linearly independent and therefore span a six-dimensional subspace $R_0 \oplus R_1 \oplus R_2 \oplus \ldots \oplus R_5$, where the symbol $\oplus$ denotes a direct sum. The linear independence is shown by taking a linear combination of the six row vectors and setting it to zero. One obtains a lower triangular matrix of the coefficients that can all be shown to be zero, proving the linear independence of the row vectors. B_1 is the domain of the map ψ. Now we come to the range of ψ, which is the first block B_1^{trans} of V^{trans}. B_1^{trans} also consists of six linearly independent row vectors, because the inverse $[V^{\mathrm{trans}}(\alpha)]^{-1} = V^{\mathrm{trans}}(-\alpha)$, in principle, always exists—we can always undo the delay by reversing the situation. The existence of the inverse implies that the row space of B_1^{trans} must be six-dimensional. This proves that the map ψ is bijective where $\psi(B_1) = B_1^{\mathrm{trans}}$. This map can be extended in an obvious way to the rest of the blocks of V and V^{trans}, hence establishing formally the correspondence as desired. This proves the existence of ψ^{-1}.

The isomorphism can also be proved explicitly by computing the translation matrices. We show this only for the block B_1. The argument extends in an obvious way to V. However, before we apply the translation matrices, we need to shift the rows of B_1 to the left by the appropriate number of columns. This is done by projecting out each row at a time by applying the projection operators $P_k, k = 0, 1, \ldots, 5$ and then shifting to the left by the required number of columns. The matrices P_k are 6×6 and have all entries 0, except for 1 on the kth row and column. Applying P_k on the left of B_1 picks out the kth row, zeroing out other rows. Then we need to shift the rows by k columns to the left. This is achieved by applying shift matrices S_k, which are 11×6. These are nothing but essentially the delay matrices D_k for integer valued delays. Finally, we apply the translation matrices T_k, which map a row vector $L_0 \to L_k$, where we define the row vector $L_k = [l_0(\alpha+k), l_1(\alpha+k), \ldots, l_5(\alpha+k)]$. Thus, we write $L_k = L_0 T_k$. The T_k are 6×6 matrices. We therefore obtain

$$B_1^{\mathrm{trans}} = \sum_{k=0}^{5} P_k B_1 S_k T_k. \tag{3.7}$$

This is the map ψ.

We can also invert the above relation. We only need to multiply the above sum by the projection operator P_j from the left, because $P_j P_k = P_j \delta_{jk}$. This picks out the jth term, zeroing out the other terms. Thus, we obtain

$$B_1 = \sum_{k=0}^{5} P_k B_1^{\mathrm{trans}} T_k^{-1} S_{-k}. \tag{3.8}$$

This is the inverse map ψ^{-1}. Here S_{-k} is the operator that shifts the elements to the right by k columns and undoes the effect S_k. Further, the T_k matrices are invertible—in fact, $\det[T_k] = 1$. We can compute them explicitly. We can write any $l_n(\alpha+k)$ as linear combinations of $l_j(\alpha)$. For example, $l_0(\alpha+1) = -l_5(\alpha), l_1(\alpha+1) = l_0(\alpha) + 6 l_5(\alpha), \ldots.$ This equation can be written in matrix form: $L_1 = L_0 T_1$, where

$$T_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 6 & -15 & 20 & -15 & 6 \end{bmatrix}. \tag{3.9}$$

The other translation matrices T_k can be obtained easily from the addition theorem for Lagrange polynomials. All the translation matrices T_k are nonsingular and, in fact, have determinant 1. This is because the vectors are rigidly translated, keeping the volume of the parallelepiped defined by those vectors invariant.

Thus, the isomorphism can be explicitly established directly.

IV. THE GENERALIZATION TO TWO ARMS

In this section, we indicate how to generalize to the case of two unequal arms. Here we have, in general, two different delays α_1 and α_2 corresponding to the arms 1 and 2, respectively. The mathematical structure is that of the product ring. We will do this for a group $\mathcal{G}$. The product is denoted by $\mathcal{G} \times \mathcal{G}$. It is defined as follows.

Let $\mathcal{G}$ be a group and let $g_1, g_2 \in \mathcal{G}$, then the element of $\mathcal{G} \times \mathcal{G}$ is the ordered pair (g_1, g_2) or we write $(g_1, g_2) \in \mathcal{G} \times \mathcal{G}$. The composition law in $\mathcal{G} \times \mathcal{G}$ is defined in the following way: Let (g_1, g_2) and (h_1, h_2) belong to $\mathcal{G} \times \mathcal{G}$, then $(g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, g_2 h_2)$. Clearly the product so defined is in $\mathcal{G} \times \mathcal{G}$. It is easily shown that under this composition law $\mathcal{G} \times \mathcal{G}$ is a group.

The next point to consider is a matrix representation of $\mathcal{G}$, which associates a matrix T_g with each element $g \in \mathcal{G}$. The matrices T_g are actually linear maps from a vector space $T_g : W \to W$. A representation is a homomorphism ϕ, which takes $g \in \mathcal{G}$ to T_g or $\phi(g) = T_g$, such that for all $g, h \in \mathcal{G}$, $\phi(gh) \equiv T_{gh} = T_g T_h$, and $\phi(e) = I$ or the identity of the group e is mapped to the identity matrix.

From the above considerations, we may easily define a representation of $\mathcal{G} \times \mathcal{G}$. Consider a finite-dimensional representation, that is, $\dim(W) = N$. Then T_g is an $N \times N$ matrix. Now consider an element $(g, h) \in \mathcal{G} \times \mathcal{G}$, then we have the corresponding $N \times N$ matrices T_g and T_h. We define the product representation $\phi \otimes \phi$ by the block diagonal $2N \times 2N$ matrix,

$$\phi \otimes \phi[(g, h)] = \begin{bmatrix} T_g & 0 \\ 0 & T_h \end{bmatrix}. \tag{4.1}$$

It is easy to show that $\phi \otimes \phi$ constitutes a representation of $\mathcal{G} \times \mathcal{G}$. It is important to note that the two block matrices are essentially independent of each other.

For the two-arm case there are, in general, two independent time delays, say α_1 and α_2 (these are the g and h in the above discussion). Under the representation homomorphism they map to 6×6 matrices $B_1^{\mathrm{trans}}(\alpha_1)$ and $B_1^{\mathrm{trans}}(\alpha_2)$, respectively, or if one considers the entire matrices $V^{\mathrm{trans}}(\alpha_1)$ and $V^{\mathrm{trans}}(\alpha_2)$. Under the isomorphism ψ, V^{trans} matrices map to the V matrices. These can be arranged as $N \times N$ block diagonal matrices as in Eq. (4.1) to obtain the product representation matrix, which is $2N \times 2N$. Then, by the following matrix transformation below, this block diagonal matrix is converted into a $2N \times N$ matrix $V_{\text{two arms}}$,

$$\begin{bmatrix} V(\alpha_1) & 0 \\ 0 & V(\alpha_2) \end{bmatrix} \begin{bmatrix} I_N \\ I_N \end{bmatrix} = \begin{bmatrix} V(\alpha_1) \\ V(a_2) \end{bmatrix} \equiv V_{\text{two arms}}, \tag{4.2}$$

where I_N is the $N \times N$ unit matrix. This is not an isomorphism.

Finally, the design matrix M (except for the subtraction of the identity matrix) is obtained by a permutation of the rows of $V_{\text{two arms}}$. In [18] the design matrix M is so constructed that the data y are interleaved with the odd numbered rows being the measurements y_1 from arm 1 and even numbered rows being the measurements y_2 from arm 2. The structures of the column vector of the measurements y and the design matrix M are displayed in Eq. (2.1). So in order to get $V_{\text{two arms}}$ into the interleaved form, we must permute its rows as required in this scheme. The scheme is as follows. Let v_j be the rows of $V_{\text{two arms}}$ and v'_j be the rows of the interleaved matrix, say $V_{\text{interleave}}$, where $j = 1, 2, \ldots, 2N$. Then we must have $v_1 \to v'_1$, $v_{N+1} \to v'_2$, $v_2 \to v'_3$, $v_{N+2} \to v'_4, \ldots$, and so on, or more generally, $v_m \to v'_{2m-1}$ and $v_{N+m} \to v'_{2m}$, where $m = 1, 2, \ldots, N$. This is a permutation of the rows of $V_{\text{two arms}}$. It can be represented by a matrix $P_{\text{interleave}}$ as follows:

$$P_{\text{interleave}} = \begin{cases} \delta_{2m-1,m} \\ \delta_{2m,N+m}. \end{cases} \quad m = 1, 2, \ldots, N. \tag{4.3}$$

$P_{\text{interleave}}$ is a $2N \times 2N$ matrix, with its odd numbered rows given by the top expression on the rhs of Eq. (4.3) and even numbered rows given by the bottom expression on the rhs. Every entry of $P_{\text{interleave}}$ is either 0 or 1 with each row and column containing just one 1 and the rest zeros. To fix ideas, let us take $N = 3$, for example, then $P_{\text{interleave}}$ is the 6×6 matrix

$$P_{\text{interleave}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.4}$$

$P_{\text{interleave}}$ is nonsingular and hence also an isomorphism. We then obtain

$$V_{\text{interleave}} = P_{\text{interleave}} V_{\text{two arms}}. \tag{4.5}$$

Thus, we have an intimate relation between the design matrices M of [18] and the matrices V^{trans} that form a representation of the delay operators.

V. CONCLUDING REMARKS

Future space-based gravitational wave interferometers will rely on the use of TDI to achieve their baseline sensitivities. The matrix representations of the TDI delay operators discussed in this article should simplify and make the implementation of TDI more efficient and, consequently, the analysis of gravitational wave signals that we are searching for. In the process, we have made a detailed analysis of the design matrices. In this article, we have shown that the matrix representation of the delay operators derived in [21] is isomorphic (i.e., one-to-one and onto) to the matrices introduced in [18] to cancel the laser noise. The isomorphism we have just established should help us in identifying a systematic way for relating the laser noise-free combinations identified by TDI to those obtained by the method proposed in [18]. This will be the subject of a forthcoming investigation.

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APPENDIX: THE INTERTWINING MAP $\tilde{\psi}$

Although the map ψ exhibits closure property and associativity, it does not map the identity to identity because B_1 is not a square matrix. This is easily remedied by suitably augmenting the matrices by identity and zero matrices. The map ψ is extended to $\tilde{\psi}$ as follows. In Eqs. (3.7) and (3.8) we make all matrices 11×11. In the projection matrix P_k we let $k = 0, 1, \ldots, 10$ and call it $\tilde{P}_k$. We also define $\tilde{B}_1$, $\tilde{S}_k$, and $\tilde{T}_k$ by appropriately adjoining 5×5 identity matrices and zero matrices as follows:

$$\tilde{B}_1^{\rm trans} = \begin{bmatrix} B_1^{\rm trans} & 0 \\ 0 & I_5 \end{bmatrix}, \qquad \tilde{B}_1 = \begin{bmatrix} & B_1 \\ 0 & I_5 \end{bmatrix}, \qquad \tilde{S}_k = \begin{bmatrix} S_k & 0 \\ & I_5 \end{bmatrix}, \qquad \tilde{T}_k = \begin{bmatrix} T_k & 0 \\ 0 & I_5 \end{bmatrix}, \tag{A1}$$

where I_5 is a 5×5 identity matrix and the block matrices 0 have appropriate dimensions to make all the matrices 11×11. Then the corresponding quantities in Eqs. (3.7) and (3.8) can be replaced by the quantities with tildes, and the sum over k over 6 terms is replaced by a sum over 11 terms, $k = 0, 1, 2, \ldots, 10$. It is now easy to check that $\tilde{\psi}$ not only satisfies the properties of ψ but also maps identity to identity; that is, $\tilde{\psi}(I_{11}) = I_{11}$, where I_{11}, the 11×11 identity matrix.

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