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ABSTRACT

We analyze how the structure of complex networks of non-identical oscillators influences synchronization in the context of the Kuramoto model. The complex network metrics assortativity and clustering coefficient are used in order to generate network topologies of Erdös–Rényi, Watts–Strogatz, and Barabási–Albert types that present high, intermediate, and low values of these metrics. We also employ the total dissonance metric for neighborhood similarity, which generalizes to networks the standard concept of dissonance between two non-identical coupled oscillators. Based on this quantifier and using an optimization algorithm, we generate Similar, Dissimilar, and Neutral natural frequency patterns, which correspond to small, large, and intermediate values of total dissonance, respectively. The emergency of synchronization is numerically studied by considering these three types of dissonance patterns along with the network topologies generated by high, intermediate, and low values of the metrics assortativity and clustering coefficient. We find that, in general, low values of these metrics appear to favor phase locking, especially for the Similar dissonance pattern.

The topology of networks of phase oscillators plays a very important role on the synchronization of the system. The individual dynamics of each oscillator, characterized by their individual frequencies, also play a very important role, which is not completely understood. What effect the emergency of cycles, the connection of nodes with close or very distinct degree have on synchronization? Furthermore, is this affected by the natural frequencies of the oscillators being connected? These questions are also important if we take into consideration the emergence of synchronization phenomena in nature that leads the involved agents from the disorder to order in a scenario in which the agent interconnections are not all-to-all. Here, we investigate these issues.

I. INTRODUCTION

Synchronization is a process in which dynamical systems manage to coordinate some dynamical properties by being connected among themselves or by being driven by a common force.[1] It is a universal behavior that takes place in many natural and artificial multi-agent systems.[2–7] In order to study synchronization in systems of interacting dynamical units, it has been shown to be useful to describe a system as a complex network of interacting oscillators,[8,9] where nodes represent the dynamical units and the connections between them express their interacting mechanisms, where nodes only interact with adjacent units. One of the most widely used paradigmatic models of phase oscillators to study synchronization in complex networks is the Kuramoto model.[10,11]

A great number of natural phenomena where a system is composed of interconnected dynamical units can be modeled by using complex networks to capture its global and emerging properties.[12–14] For example, it can be used in the study of non-linear dynamical systems,[15,16] of chemical and biological systems,[17–21] of power grids,[22–25] and even in the study of social networks.[26–28] The synchronization of networks in a multi-layer network has also been the subject of intense studies and can be used, for example, in the study of epidemic models.[29,30]

Biological and social studies have shown that in some situations, like when it comes to choosing friends, people prefer to gather with similar minded ones.[31–34] On the other hand, when it comes to mating preferences, some species prefer to mate with dissimilar ones, which may provide the offspring with good genes.[35–38] Freitas *et al.*[26] used an approach based on an interconnected network of Kuramoto oscillators to analyze these scenarios. There, the former kind of behavior is referred to as *Similar* ($\mathscr{S}$), while the latter one as *Dissimilar* ($\mathscr{D}$) neighborhood patterns. If an ensemble presents no strong bias toward any of these extremes, it is called *Neutral* ($\mathscr{N}$).

In this work, we explore the idea of Similarity and Dissimilarity described above by means of structure properties and synchronization of complex networks of Kuramoto non-identical phase oscillators. In order to quantify these patterns, we shall use a measure related to classical *dissonance*,[1] which measures the difference of the natural frequencies of a pair of oscillators.

With reference to related material on synchronization of complex networks, Pinto and Saa[39] employs a dimensional reduction approach proposed by Ref. 40 and derive a sufficient analytical condition, considering an ansatz, to optimize a topology of a network in order to favor synchronization using the Kuramoto model. They also showed that when this method is applied to a network with random natural frequencies, the final topology presents a negative correlation between the natural frequencies of adjacent vertices in a way that we can call a network with a Dissimilar pattern, even though the approach in Ref. 39 does not exhaust the problem, especially for small and intermediate coupling values, which are commonly found in nature.[1] A numerical study made by Freitas *et al.*[26] showed that Similar patterns favor weaker forms of synchronization, but Dissimilar ones exhibit explosive synchronization, reaching global synchronization faster than the Similar pattern.[41] We use an evolutionary strategy to find a minimal network structure that guarantees global synchronization and show that the heterogeneity in the nodes' natural frequency is the driving force that determines the evolution of the network structure.

We intend to extend the work done by Ref. 26 and add the complex network measures assortativity and clustering coefficient to investigate how the structure of complex networks influences the synchronization of Similar, Dissimilar, and Neutral patterns of natural frequencies of oscillators. Assortativity is employed in order to measure how connections between nodes with the same degree influence the emergence of synchronization, while the clustering coefficient measures the impact of loops of size three (small cycles). Therefore, the topology of the networks is dictated by the assortativity and by the clustering coefficient values, while the natural frequency of their nodes is given by the dissonance patterns.

The authors in Refs. 42 and 43 used a modified version of the Kuramoto model in order to study opinion formation and its dynamics through synchronization of complex networks where the phase of a node in this model represents the opinion of an individual and the coupling represents the amount of the interaction among them. To illustrate the meaning of the metrics used here, the natural frequency patterns, and the synchronization of the network, let us take as an example a large group of individuals having an argument about a polemic subject where each individual has its own initial opinion, and due to the number of people and the limited time they have, they can only communicate with a limited number of people inside this group. Their discussion ends only when all participants come to an agreement and, therefore, reach a common opinion. We can model this situation by using a complex network approach where each individual is represented by a node whose behavior is dictated by a dynamical system model, the interaction between them is represented by an edge, and the opinion of each individual in relation to the subject being discussed is given by the natural frequency of the nodes. Therefore, reaching a common opinion is associated with a synchronized state. The natural frequency patterns Similar, Dissimilar, and Neutral here relate to the level of homogeneity (Similar) or heterogeneity (Dissimilar) of the opinion of communicating individuals, as, for example, if the individuals only communicate with similar minded ones, the Similar pattern is used to model this dynamics. We also refer the reader to Noorazar[44] and Deffuant *et al.*[45] for a more detailed discussion on opinion dynamics.

The rules of who can communicate with whom are given by the metrics assortativity and the clustering coefficient. If individuals who interact with many people prefer to communicate with the ones that are also popular and individuals who interact with a few people prefer to communicate with ones that are also less popular, the network is said to be assortative and has a high value of the metric assortativity. The opposite can also happen; when popular individuals tend to talk with less popular ones, the network is said to be disassortative. Looking at another aspect of the rules of communication within this group of people, we can also allow two contacts of a person to talk to each other, forming then a small cycle or a loop of size three in the network topology. When there is a large number of a couple of contacts of individuals communicating to each other, we say that this network presents a high clustering coefficient, and, on the other hand, it presents a low clustering coefficient if the opposite happens.

In this scenario, one can ask the following: how strong the interactions (represented here by the coupling of the Kuramoto model) between individuals must be in order to reach an agreement? Is it easier to be achieved if individuals only communicate with similar minded ones or is it the opposite? Is it easier if popular individuals only talk to each other or when they talk to less popular ones? Or if contacts of individuals communicate with each other?

Our results show that the Similar pattern of a natural frequency distribution favors weaker forms of synchronization, but, as we increase the coupling constant, the Dissimilar pattern is the first to reach the synchronized state. The Erdös–Rényi model presented itself as the easiest to reach the phase locking state when compared to Watts–Strogatz and Barabási–Albert network models. In relation to the network metrics assortativity and clustering coefficient, one can see that low values of both metrics favors the reaching of the synchronized state. As for the questions raised about the best strategy to conduct an argument among a group of people, we find that the best strategy would be to encourage individuals with different opinions to communicate to each other and, at the same time, encourage popular individuals to talk to less popular ones and discourage the interaction among contacts of individuals in order to avoid small cycles of interactions.

This paper is organized as follows: Sec. II presents the Kuramoto model, characterize the Similar, Dissimilar, and Neutral frequency patters, defines synchronization quantifiers, and presents the network metrics used to generate the network topologies. Section III develops a discussion about the results, and the conclusions are presented in Sec. IV.

II. MODEL AND METHODS

In this work, we consider complex networks of Kuramoto phase oscillators whose dynamics is described by a simple but very powerful model as it has proven to accurately approximate a great class of coupled oscillators.[46,47] The dynamics of the Kuramoto model is described by

$$\dot{\theta}_i = \omega_i + \frac{\lambda}{d_i}\sum_{j=1}^{N} A_{ij}\sin(\theta_j - \theta_i), \tag{1}$$

where N is the number of oscillators, $\theta_i \in \mathbb{R}$ is the phase variable of each oscillator for $i = 1, \ldots, N$, and ω_i is its natural frequency. Communication channels are defined through a coupling graph, a simple and connected graph, which is expressed via its adjacency matrix (A_{ij}); i.e., A_{ij} has value 1 if nodes i and j are connected and 0 otherwise. The symbol d_i denotes the node degree of the ith oscillator, while $\lambda \geq 0$ is the overall coupling constant.

In order to characterize the Similar, Dissimilar, and Neutral natural frequency patterns on complex networks, we make use of the total dissonance measure[26]

$$\nu = \frac{1}{N}\sqrt{\sum_{i,j=1}^{N} A_{ij}(\omega_i - \omega_j)^2}. \tag{2}$$

For the Similar pattern, the natural frequencies of adjacent nodes are close to each other such that the value of ν is small and it is zero only if all oscillators have identical natural frequencies. If the natural frequencies of adjacent nodes are very different from each other, that is, the Dissimilar pattern, the value of ν is higher. The Neutral pattern is characterized as intermediate values of ν. To calculate these frequency patterns, the stochastic optimization algorithm called simulated annealing[48] is used. In order to optimize the objective function ν, it makes permutations of the natural frequencies set until it finds an optimal local value of the objective function, in correspondence with the desired Similar or Dissimilar patterns. Considering the outputs of this algorithm, the minimization of ν corresponds to the Similar pattern, the maximization to the Dissimilar one, and the random initial natural frequency set is called Neutral. In practice, for each network topology considered in this work, a set of natural frequencies is chosen from a random uniform distribution in $[-\pi, \pi]$ and the total dissonance ν_{ini} is calculated, this one is called the Neutral frequency pattern. Then, an optimization algorithm is applied in order to maximize (ν_{max}) and minimize (ν_{min}) the total dissonance of each network, giving rise to the Dissimilar and Similar patterns, respectively.

A useful way to quantify phase synchronization of networks is by using the order parameter R defined as

$$R(t) = \left|\frac{1}{N}\sum_{i=1}^{N} e^{i\theta_i}\right|, \tag{3}$$

where $R(t) \in [0, 1]$ measures the amount of collective behavior of the system. When $R(t) = 1$, the system is said to be in the state of phase synchronization and all oscillators present the same phase. On the other hand, when the system presents an incoherent behavior, $R(t) \approx 0$.

As done by Ref. 49, we introduce now an index to quantify the appearance of another type of synchronization, called *phase locking* (PL) that indicates when a pair of oscillators presents a constant phase difference and, therefore, moves as a rigid body. This measure is called *partial synchronization index* S_{ij} given by

$$S_{ij} = \left|\lim_{\Delta t\to\infty}\frac{1}{\Delta t}\int_{t_r}^{t_r+\Delta t} e^{i[\theta_i(t)-\theta_j(t)]}\,dt\right|, \tag{4}$$

where $S_{ij} \in [0, 1]$ and t_r is a large enough transient time. When two oscillators have the same instantaneous frequency, they are said to be in phase lock and, in this case, S_{ij} is equal to 1. In order to measure the degree of partial synchronization of the whole network, we calculate the arithmetic mean

$$S = \frac{1}{N^2}\sum_{i,j=1}^{N} S_{ij}. \tag{5}$$

Therefore, when the whole system is in phase lock, that is, when the phase difference between all pair of nodes is constant in time, $S = 1$. As for the order parameter in this case, $R(t)$ is constant in time but not necessarily equal to 1 as it is not mandatory that the phases are the same. If $S \approx 0$, the system presents a low coherent behavior.

Our aim is to analyze how network structure influences the emergency of synchronization on complex networks. For this purpose, the network measures called assortativity and clustering coefficient are used in order to generate different network topologies.

Assortativity measures the similarity of connections in a network with respect to a certain characteristics of a node. In this work, the assortativity is determined by the degree of the nodes, and it is given by the use of the Pearson correlation coefficient[50,51]

$$\rho = \frac{\sum_{ij} ij(f_{ij} - a_i b_j)}{\sigma_a \sigma_b}, \tag{6}$$

where a_i and b_j are the fraction of edges that start and end at nodes with degree values i and j, respectively, f_{ij} is the fraction of edges between nodes of degree i and j, and σ_a and σ_b are the standard deviations of the distributions a and b, respectively. a_i, b_j, and f_{ij} satisfy the sum rules: $\sum_{ij} f_{ij} = 1$, $\sum_j f_{ij} = a_i$, and $\sum_i f_{ij} = b_j$.

The graph assortativity $\rho \in [-1, 1]$ represents how nodes in a network associate with each other; i.e., it shows whether nodes prefer to connect to nodes of the same sort or of opposing sort. When, on average, high degree nodes connect to high degree ones or low degree nodes connect to low degree ones, ρ is close to 1 and the network is said to be *assortative*. On the other hand, if on average, high degree nodes connect to low degree ones, ρ is close to -1 and the network is said to be *disassortative*. If ρ is close to 0, the connections are considered to be completely random.[51] The reader should notice that there are two different mechanisms of preferential attachment here: assortativity takes into account only the node degree, while neighborhood patterns consider both graph structure and node's natural frequency.

Another basic network measure that is used in this work is the *clustering coefficient*, which measures the presence of loops of size three inside a network; i.e., it measures the tendency of two neighbors of a certain vertex to also be connected to each other. In a real world network, it can be seen as the likelihood of friends of a certain person also to be friends with each other.[52] The clustering coefficient of a vertex is given by

$$c_i = \frac{2T_i}{d_i(d_i-1)}, \tag{7}$$

where T_i is the number of triangles involving node i and d_i is the degree of node i. Therefore, the clustering of a node $c_i \in [0,1]$ is the number of triangles that pass through that node normalized by the maximum number of such triangles in a way that if none of the neighbors of node i are connected to each other, $c_i = 0$, and $c_i = 1$ if all neighbors are connected.[53] The average clustering coefficient of the network is given by

$$C = \frac{1}{N}\sum_{i=1}^{N} c_i. \tag{8}$$

A large clustering coefficient indicates that there are many redundant paths in the network and a low clustering indicates the opposite.

The models of complex networks analyzed in this work are Erdös–Rényi (ER),[54] Watts–Strogatz (WS),[27] and Barabási–Albert (BA)[55] as they are widely used in the literature.[8,56] For the BA model, the degree exponent is fixed as $\gamma = 3$. In the ER model, we set the probability of edge creation to be 0.15, and for the WS networks, the probability of rewriting each edge is 0.2. The number of nodes is fixed as $N = 50$.

The main contribution of this work is to analyze the impact on synchronization considering the assortativity and clustering coefficients in association with neighborhood patterns ($\mathscr{S}/\mathscr{N}/\mathscr{D}$). To do so, we proceed as follows.

Network configurations considered here are represented by the pair (A, ω), where A stands for the adjacency matrix of the graph and ω is the set of natural frequencies. For each network model (BA, ER, WS), the three corresponding network topologies are considered for assortativity and clustering (representing low, intermediate, and high values of each) $A^{\rho_{min}}, A^{\rho_{middle}}, A^{\rho_{max}}, A^{C_{min}}, A^{C_{middle}} A^{C_{max}}$, and three patterns of the distribution of natural frequencies are considered: Neutral $\omega^{\mathscr{N}}$, Similar $\omega^{\mathscr{S}}$, and Dissimilar $\omega^{\mathscr{D}}$. In all, 27 configurations are studied for assortativity and 27 for the clustering coefficient. As an example, consider a BA network with low value of assortativity $A^{\rho_{min}}$. For this network, a random set of natural frequencies is generated from a uniform distribution (Neutral dissonance pattern), giving rise to the Configuration $(A^{\rho_{min}}, \omega^{\mathscr{N}})$. Then, the simulated annealing algorithm is used to optimize the values of the total dissonance with a low value, giving rise to the set of natural frequencies of the Similar pattern and the configuration $(A^{\rho_{min}}, \omega^{\mathscr{S}})$ and a high value generating the set of natural frequencies of the Dissimilar pattern and the Configuration $(A^{\rho_{min}}, \omega^{\mathscr{D}})$. Recall that dissonance patterns do not alter the physical configuration of networks, it only interchanges the natural frequencies. The low and high assortativity/clustering values are the only ones that come from a different network configuration. The choosing of $A^{\rho_{min}}$, $A^{\rho_{middle}}$, $A^{\rho_{max}}$, $A^{C_{min}}$, $A^{C_{middle}}$, and $A^{C_{max}}$ is discussed in Sec. III.

In order to measure how the total dissonance combined with assortativity and clustering coefficients affect the global synchronization of the networks, the Kuramoto model [Eq. (1)] is numerically integrated and the mean value of the order parameter is calculated $R(t)$ over the integration time and is denoted by $\langle R\rangle$. We call $\langle R\rangle_{PL}$ and λ_{PL} the values of the order parameter and the coupling constant, respectively, at the emergence of phase locking ($S = 1$). The initial conditions are the same for all networks used in this work and were all set as $\theta_i(0) = 0.5$ for $i = 1, \ldots, N$, where N is the total number of nodes. This choice was intentional because as shown in previous works,[3] the set of initial conditions can also play an important role in the synchronization of the system, but this is not the scope of this work. The distribution of the natural frequencies for the Neutral patterns is drawn randomly by a uniform distribution over $[-\pi, \pi]$.

III. RESULTS AND DISCUSSION

ER, WS, and BA topology models are used in this work. Each of them has specific topology, and in order to obtain networks with low and high values of assortativity and clustering coefficient, we chose to create 1×10^6 networks of each type and pick three of each model, which present lowest, intermediate, and highest values of the measures being considered. In this way, we make sure to keep the topology of the network models. The histograms of all networks generated as a function of assortativity and clustering coefficients can be seen in Fig. 1.

By construction, the BA model has a preferential attachment rule when building the graph; therefore, the probability of a new node to connect with an existing one is proportional to the existing node degree. Therefore, these networks are characterized by having a few nodes highly connected (called *hubs*) and the rest of the nodes with few connections. It is by construction a network with a negative value of assortativity where nodes with low degree tend to connect to the ones with high degree. On the other hand, ER and WS do not have a preferential attachment rule, and the vertices have a rather random pattern of connections. Therefore, the average assortativity is expected to be around zero. When it comes to the clustering coefficient, the WS model is the one expected, in average, to have the higher number of loops of size three as it is constructed by rewriting some edges of a regular network, which are known to have a high clustering coefficient.[51,53,56]

We then pick the adjacency matrices A that generate extreme values of ρ and C from the histogram in Fig. 1 (ρ_{min} and C_{min} are the smallest and ρ_{max} and C_{max} are the greatest values) and ones that generate values approximately in the middle of them (ρ_{middle} and C_{middle}). Therefore, we have the BA model with $\rho_{min} = -0.7354$, $\rho_{middle} = -0.2898$, $\rho_{max} = 0.1034$; the ER model

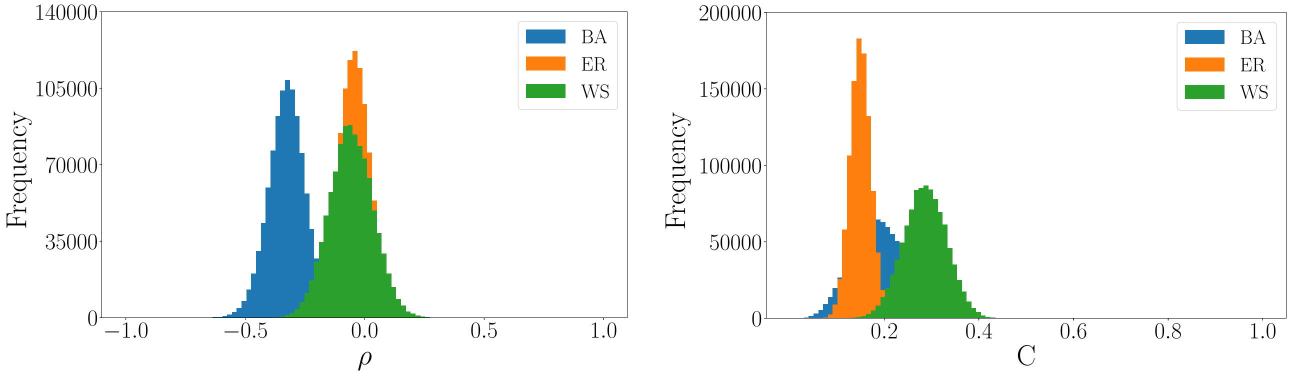

FIG. 1. Histograms of Erdös–Rényi (orange), Watts–Strogatz (green), and Barabási–Albert (blue) networks in relation to (a) assortativity (ρ) and (b) clustering coefficient (C). 1×10^6 networks were generated to compute each histogram.

with $\rho_{min} = -0.3560$, $\rho_{middle} = -0.0505$, $\rho_{max} = 0.2584$; and the WS model with $\rho_{min} = -0.5079$, $\rho_{middle} = -0.0515$, $\rho_{max} = 0.4032$.

The topologies related to the minimum and maximum values of ρ and C, along with the three dissonance patterns, can be seen in Figs. 2 and 3. One can notice that for the Similar pattern, nodes tend to be connected to ones that have a similar natural frequency (similar node color) and that for the Dissimilar pattern, they tend to be connected with nodes with different natural frequencies. This is expected; therefore, we can confirm that our optimization algorithm is working (the algorithm used to generate these patterns converges to a local, not global, value of the objective function that it is trying to maximize/minimize). The Neutral pattern stays in the middle as some nodes connect with nodes with similar frequencies and some connect with nodes with dissimilar frequencies. Recall that dissonance patterns do not alter the physical configuration of the networks, it only interchanges the natural frequencies. The low and high assortativity/clustering values are the only ones that come from a different network configuration.

The mean of the order parameter $\langle R \rangle$ and the total partial synchronization index S as a function of the overall coupling for networks with high and low assortativity and clustering coefficient and all three dissonance patterns for the BA, ER, and WS models are presented in Figs. 4 and 5, respectively. In relation to the patterns Similar, Neutral, and Dissimilar, one can note that, for small coupling, the Similar pattern favors weaker forms of synchronization both to a phase locked state (higher value of S) and to a phase synchronized state (higher value of $\langle R \rangle$) for the BA, ER, and WS models since the growth of these measures is more protuberant at first for small values of coupling. The dissimilar pattern appears to be the harder to achieve synchronization, while the Neutral one stays in the middle. As the coupling λ is increased, the Dissimilar pattern presents a higher growth on both $\langle R \rangle$ and S and is the first of the patterns to reach phase locking. As the coupling increases even more, it is time of the Neutral pattern to reach the phase locking state, and then for greater λ, the Similar pattern also synchronizes. Therefore, the Dissimilar natural frequency distribution pattern is the one that mostly favors the achievement of the synchronized state. This behavior was also observed by Freitas *et al.*[26] and Scafuti *et al.*[41]

In relation to the illustrative example given at the beginning of the paper about the discussion of a polemic subject, we can conclude that if mostly similar minded people talk to each other, an agreement seems to be close by people making only a small effort, but at some point, the discussion somehow does not advance anymore and more effort is needed in order to reach an agreement. On the other hand, when people tend to talk with the ones that have distinct opinions, there is a huge discussion at first, and, despite the increasing effort of all individuals, it seems like an agreement is not reachable, but, after more effort is made by the individuals, a common opinion can finally be reached and all individuals arrive at the same conclusion.

Now, we investigate how the measures assortativity and clustering coefficient along with the dissonance patterns affect synchronization. In order to do this, we annotate the value of λ for which all configurations in Figs. 4 and 5 reach phase locking, and we name it λ_{PL}. This result is presented in the first column in Figs. 6 (related to assortativity) and 7 (related to clustering). In the second column, there is the value of the order parameter (R_{PL}) for this λ_{PL}. The order parameter R_{PL} shows the amount of phase synchronization of the system at this stage. By definition, the partial synchronization index S at $\lambda = \lambda_{PL}$ is equal to one; therefore, the system is synchronized.

In relation to the network models considered in this work, on average, the ER model is the one that reaches phase locking with lower coupling values (light yellow) when considering the measures assortativity and clustering coefficient in Figs. 6 and 7(a), 7(c), and 7(e). The BA network topology needs on average a high coupling constant to reach the phase locking state when considering assortativity, being then the hardest to synchronize in relation to this measure. WS networks are an intermediate between these two in relation to assortativity but requires the highest values of coupling to reach phase locking when the clustering coefficient is taken into account.

In regard to the network structure, we can infer that disassortative networks seem to favor synchronization for, in general, networks with negative values of ρ require a lower coupling value in order to reach phase locking. In this way, when high degree nodes connect with low degree ones, it favors synchronization (this does

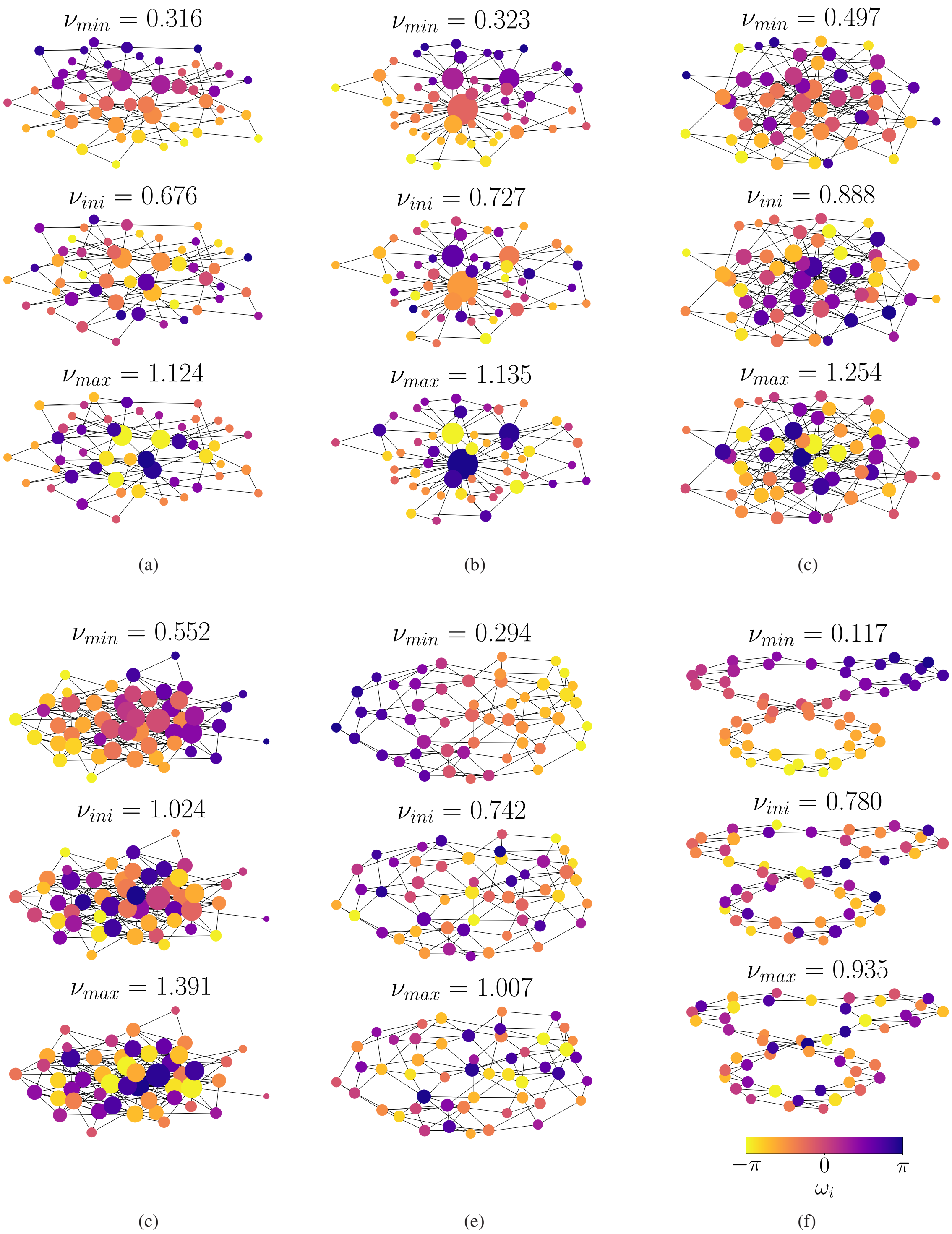

FIG. 2. (a) and (b) BA, (c) and (d) ER, and (e) and (f) WS networks with low and high values of clustering coefficient C. The Similar $\mathscr{S}$ (ν_{min}), Neutral $\mathscr{N}$ (ν_{ini}), and Dissimilar $\mathscr{D}$ (ν_{max}) patterns of dissonance ν are also showed for each network (from top to bottom, respectively). ω_i is the natural frequency of the nodes, and the size of the nodes is proportional to the degree.

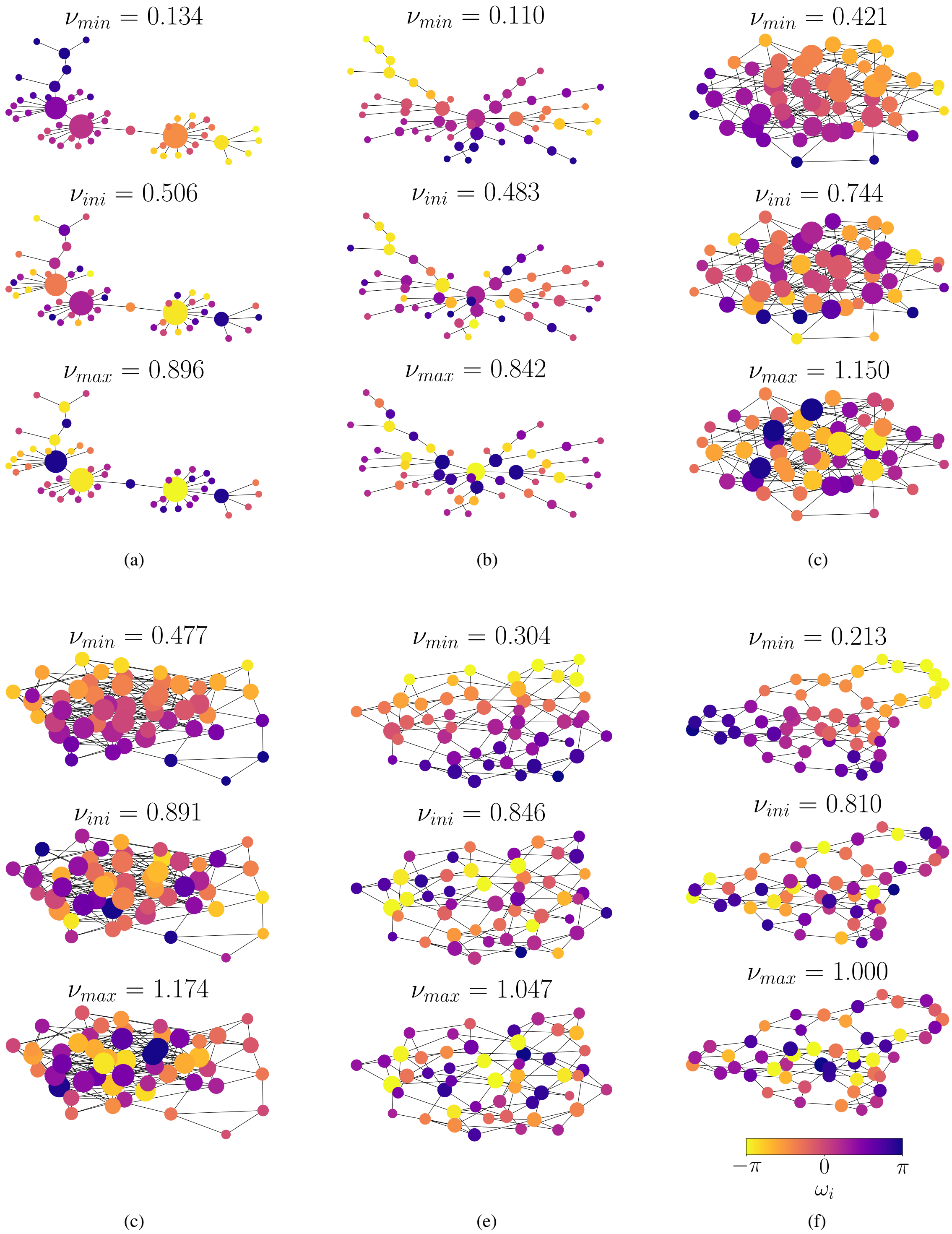

FIG. 3. (a) and (b) BA, (c) and (d) ER, and (e) and (f) WS networks with low and high values of assortativity ρ. The Similar $\mathscr{S}$ (ν_{min}), Neutral $\mathscr{N}$ (ν_{ini}), and Dissimilar $\mathscr{D}$ (ν_{max}) patterns of dissonance ν are also showed for each network (from top to bottom, respectively). ω_i is the natural frequency of the nodes, and the size of the nodes is proportional to the degree.

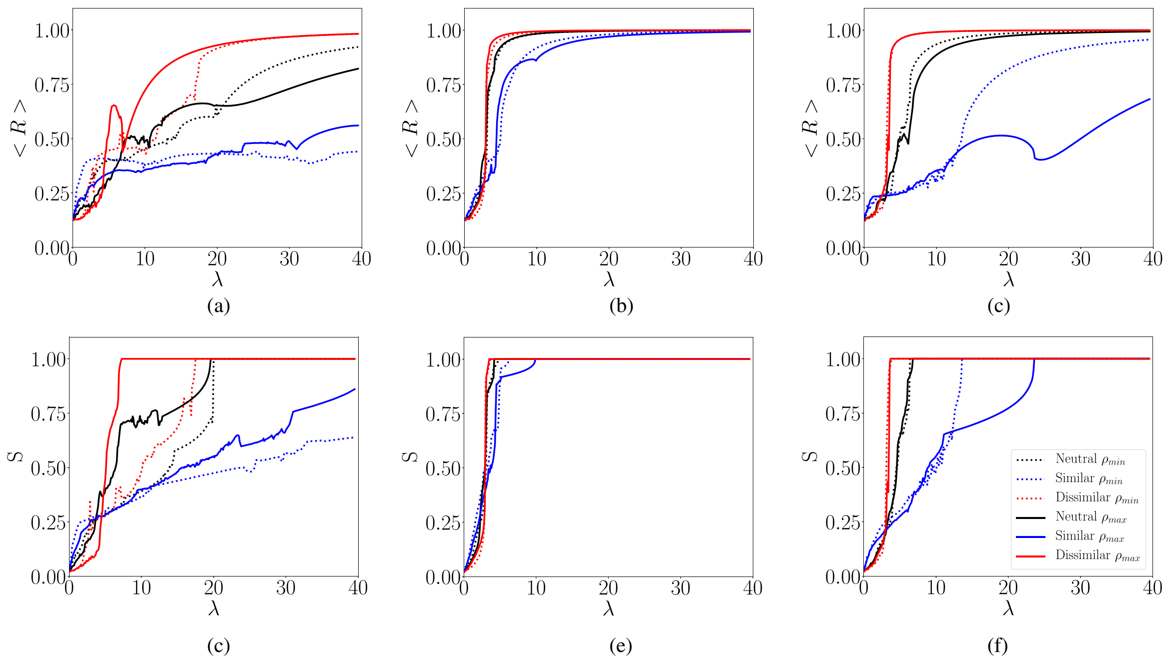

FIG. 4. (a)–(c) Mean of the order parameter $\langle R \rangle$ and (d)–(f) the total partial synchronization index S as a function of the coupling for networks with low (dashed line) and high (continuous line) values of assortativity ρ and patterns Neutral (black), Similar (blue), and Dissimilar (red) of the natural frequency distribution. Note that S converges to 1 for a finite value of λ and $\langle R \rangle$ asymptotically tends to 1.

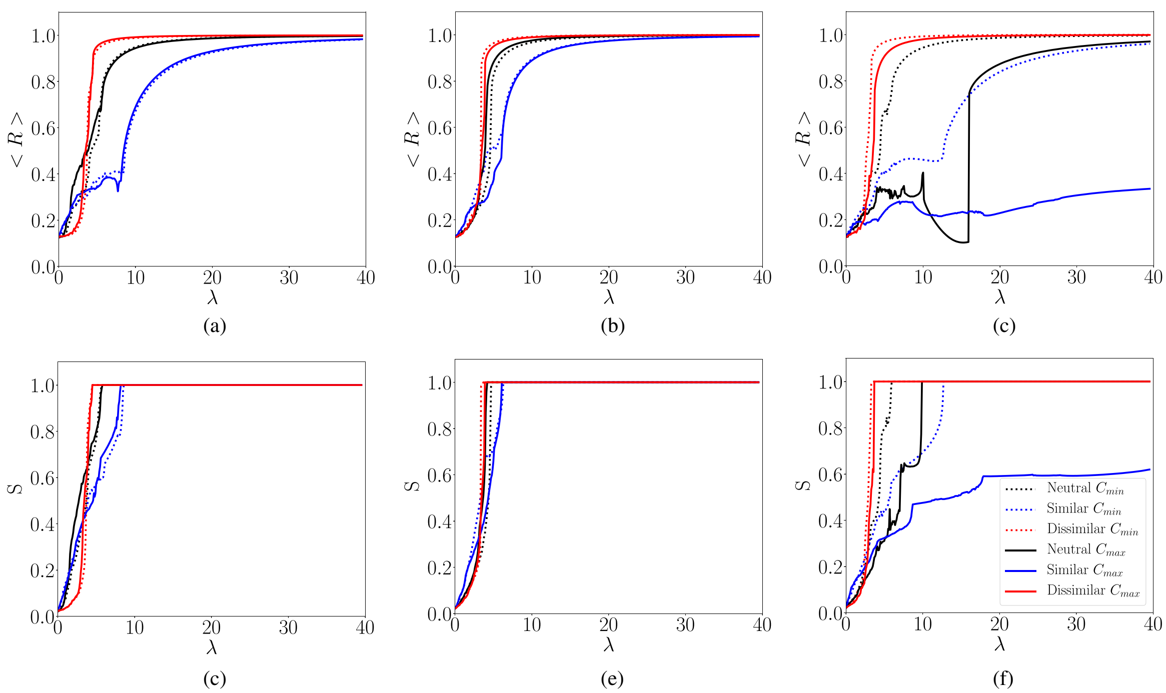

FIG. 5. (a)–(c) Mean of the order parameter $\langle R \rangle$ and (d)–(f) the total partial synchronization index S as a function of the coupling for networks with low (dashed line) and high (continuous line) values of clustering coefficient C and patterns Neutral (black), Similar (blue), and Dissimilar (red) of the natural frequency distribution. Note that S converges to 1 for a finite value of λ (except for the WS Similar C_{max}) and $\langle R \rangle$ asymptotically tends to 1.

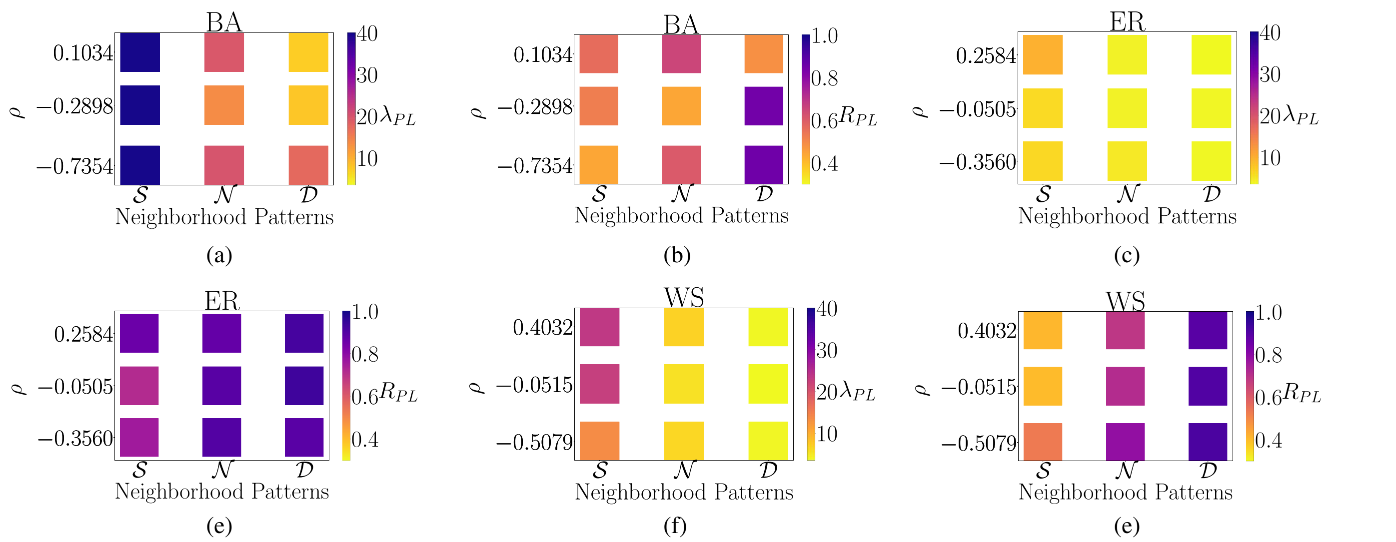

FIG. 6. Contour plot of the assortativity ρ and the neighborhood patterns in relation to the (a), (c), and (e) coupling λ_{PL} and (b), (d), and (f) order parameter R_{PL} at phase locking for the models BA, ER, and WS of networks.

not seem to apply to the BA model). As already mentioned before, the Dissimilar natural frequency pattern tends to favor synchronization, and we can think of the distribution of the nodes in a disassortative network also as being a dissimilar topological distribution as nodes with different degree tend to connect to each other. Therefore, when analyzing our example, instead of having popular individuals communicating among each other, it is best if popular individuals talk to less popular ones.

In relation to the clustering coefficient, networks with fewer loops of size three seem to favor synchronization as, on average, networks with the lowest value of C tend to be easier to synchronize. When it comes to our example, this means that it is best to avoid the

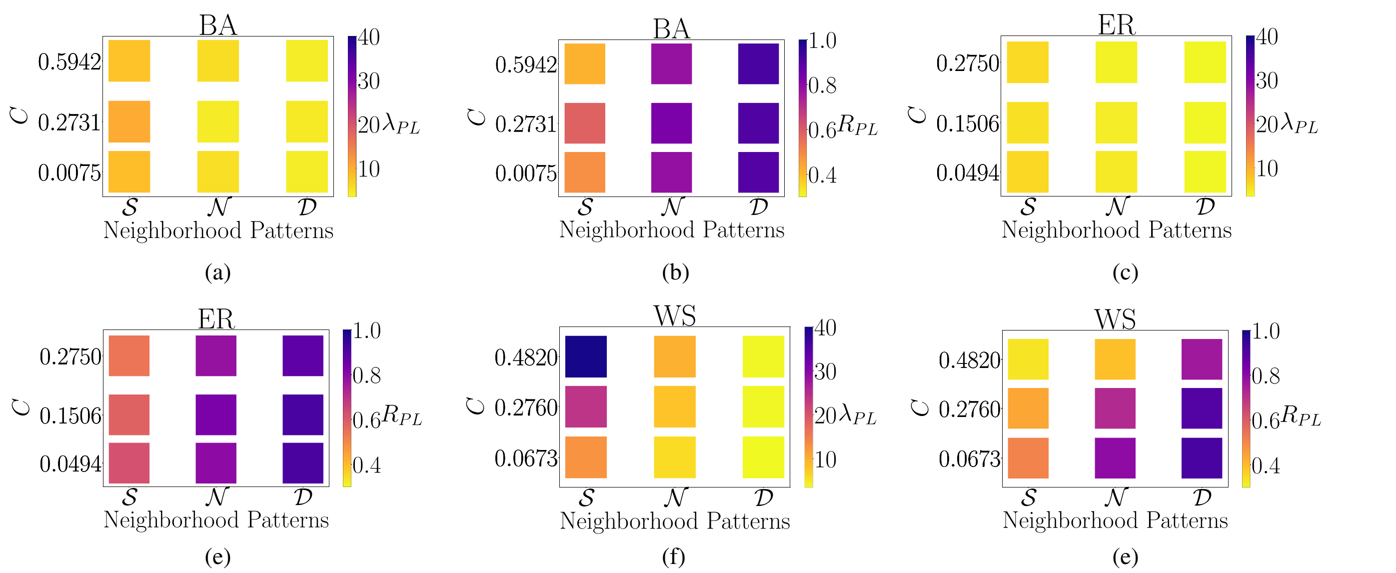

FIG. 7. Contour plot of the clustering coefficient C and the neighborhood patterns in relation to the (a), (c), and (e) coupling λ_{PL} and (b), (d), and (f) order parameter $\langle R \rangle_{PL}$ at phase locking for the models BA, ER, and WS of networks.

contacts of an individual to communicate with each other, avoiding then the creation of a small cycle of discussion as this may create unnecessary debates and, therefore, increase the effort to achieve an agreement.

In general, the measures assortativity and clustering coefficient seem to have a stronger effect on the synchronization of the Similar dissonance patterns (especially when considering the WS model), having a modest effect on the Neutral pattern and a very low effect on the Dissimilar one.

IV. CONCLUSIONS

The influence of the structure of complex networks of non-identical oscillators on global synchronization was studied. The total dissonance metric for neighborhood similarity was employed, and, with the help of an optimization algorithm, three patterns of natural frequency distributions were created, one where adjacent nodes have similar frequencies (Similar pattern), one where they have different frequencies (Dissimilar), and one that is a blend of both (Neutral). Network topologies of the models Erdös–Rényi, Watts–Strogatz, and Barabási-Albert with high, intermediate, and low values of the network measure assortativity and clustering coefficient were created and along with the frequency patterns were used to study the synchronization of these systems.

In relation to the emergency of phase locking, at low values of the coupling constant, the Similar pattern clearly favors weaker synchronization regimes, but, as the coupling is increased, the Dissimilar pattern presents a rapid growth and is the first to reach synchronization, which corroborates previous works.[26,41] As for the complex network models used in this work, the Erdös–Rényi showed itself as the easiest to reach the regime of synchronization when compared to Watts–Strogatz and Barabási–Albert, but this has yet to be confirmed by future experiments by comparing, for example, these three models where each one has the same values of assortativity and/or clustering coefficient. In relation to the network measures employed here, in general, both low values of assortativity and clustering coefficient appear to favor synchronization, especially for the Similar dissonance pattern.

In summary, answering the questions raised at the beginning of this paper, based on our findings, we can state that the best way to conduct a discussion on a polemic subject is by encouraging individuals with different opinions to talk to each other and also encourage popular individuals to talk to less popular ones. It is also a good idea to avoid contacts of individuals to talk to each other, avoiding then small cycles of discussions. This hypothesis has yet to be confirmed by future experiments.

As for future work, we consider to use the BA model with distinct degree exponents. We also intend to investigate the role that the average degree of the networks has on synchronization. The behavior of the *WS* configuration ($A^{C_{min}}, \omega^{\mathscr{S}}$), which does not reach the synchronous state even for high values of λ, as shown in Fig. 2, has also to be better analyzed.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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